Chaotic dynamics of Hamiltonian systems

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Outline

- **Chaos**
- **Autonomous Hamiltonian systems: Example Hénon-Heiles system**
- **Regular vs Chaotic motion**
- **Visualization of chaos: Poincaré Surface of Section (PSS)**
- **Chaos Indicators**
  - ✓ **Variational equations and Tangent map**
  - ✓ **Lyapunov exponents**
  - ✓ **Smaller ALignment Index – SALI**
  - ✓ **Generalized ALignment Index – GALI**
- **Efficient numerical integration methods**
  - ✓ **Symplectic integrators**
  - ✓ **The tangent map (TM) method**

Chaos

Definition [Devaney (1989)]

Let V be a set and $f : V \to V$ a map on this set.

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We say that f is chaotic on V if

1. **f has sensitive dependence on initial conditions.**
2. **f is topologically transitive.**
3. **periodic points are dense in V.**

Chaos

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$\mathbf{f}: V \rightarrow V$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $\mathbf{x} \in V$ and any neighborhood Δ of $\mathbf{x}$, there exist $\mathbf{y} \in \Delta$ and $n \geq 0$, such that $|\mathbf{f}^n(\mathbf{x}) - \mathbf{f}^n(\mathbf{y})| > \delta$, where $\mathbf{f}^n$ denotes n successive applications of **f**.

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There exist points arbitrarily close to **x** which eventually separate from **x** by at least δ under iterations of **f**.

Not all points near **x** need eventually move away from **x** under iteration, but there must be at least one such point in every neighborhood of **x**.

Chaos

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f $: V \rightarrow V$ is said to be *topologically transitive* if for any pair of open sets $U, W \subset V$ there exists $n > 0$ such that $\mathbf{f}^{n}(U) \cap W \neq \emptyset$.

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Consequently, the dynamical system cannot be decomposed into two disjoint invariant open sets.

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3. **An element of regularity because it has periodic points which are dense.**

Usually, in physics and applied sciences, people use the notion of chaos in relation to the sensitive dependence on initial conditions.

Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1, q_2, \ldots, q_N}_{\text{positions}}, \underbrace{p_1, p_2, \ldots, p_N}_{\text{momenta}})$$

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The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0),\ q_2(0),\ldots,q_N(0),\ p_1(0),\ p_2(0),\ldots,p_N(0))$$

is governed by the Hamilton's equations of motion

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Phase space: the 2N dimensional space defined by variables $q_1,q_2,\ldots,q_N,\ p_1,p_2,\ldots,p_N$

Example (Hénon-Heiles system)

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$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

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THE ASTRONOMICAL JOURNAL VOLUME 69, NUMBER 1 FEBRUARY 1964

The Applicability of the Third Integral Of Motion: Some Numerical Experiments

Michel Hénon* and Carl Heiles

Princeton University Observatory, Princeton, New Jersey

(Received 7 August 1963)

The problem of the existence of a third isolating integral of motion in an axisymmetric potential is investigated by numerical experiments. It is found that the third integral exists for only a limited range of initial conditions.

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Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

Regular vs Chaotic orbits

Hénon-Heiles system

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For H=0.125 we get a regular and a chaotic orbit with initial conditions (ICs):

$x=0$, $y=0.1$, $p_y=0$ and **$x=0$, $y=-0.25$, $p_y=0$.**

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Orbit **Perturbed**

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			Orbit	**Perturbed**
t=	100	x=	0.13299571833307644	0.13299571833372630 64

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		Orbit	Perturbed
t=	100	x= 0.13299571833307644	0.13299571833**7263064**
t=	5000	x= 0.37699928389102310	0.37699928387**0156576**

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t= 100	x=	0.13299571833307644	0.13299571833337263064
t= 5000	x=	0.37699928389102310	0.37699928370156576
t= 10000	x=	-0.15909458335685224	-0.15909458334126030 9

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t= 5000	x=	0.376999283889102310	0.376999283870156576
t= 10000	x=	-0.159094583356855224	-0.159094583341260309
t= 50000	x=	0.101992400739955760	0.101992400253961321

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		Orbit	Perturbed
t=	100	x= 0.13299571833307644	0.13299571833726306 4
t=	5000	x= 0.37699928389102310	0.37699928370156576
t=	10000	x=-0.15909458335685522 4	-0.15909458334126030 9
t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1
t=	100000	x=-0.38112053374651178 0	-0.38112053332725887 0

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t= 10000	x=-0.159094583356855224	-0.159094583341260309
t= 50000	x= 0.101992400739955760	0.101992400253961321
t=100000	x=-0.381120533746511780	-0.381120533327258870
t= 100	x= 0.090272817735167835	0.090272821355768668

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t= 100	x= 0.090272817735167835	0.090272821355768668
t= 200	x= 0.295031687482249283	0.295031884858625637

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We perturb both ICs by **$\delta p_y=10^{-11}$** (!) and check the evolution of x

	Orbit	Perturbed
t= 100	x= 0.132995718333307644	0.132995718337263064
t= 5000	x= 0.376999283889102310	0.376999283870156576
t= 10000	x=-0.159094583356855224	-0.159094583341260309
t= 50000	x= 0.101992400739955760	0.101992400253961321
t=100000	x=-0.381120533746511780	-0.381120533327258870
t= 100	x= 0.090272817735167835	0.090272821355768668
t= 200	x= 0.295031687482249283	0.295031884858625637
t= 300	x= 0.515226330109450181	0.515225440480693297

Regular vs Chaotic orbits

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	Orbit	Perturbed
t= 100	x= 0.132995718333307644	0.132995718337263064
t= 5000	x= 0.376999283889102310	0.376999283870156576
t= 10000	x=-0.159094583356855224	-0.159094583341260309
t= 50000	x= 0.101992400739955760	0.101992400253961321
t=100000	x=-0.381120533746511780	-0.381120533327258870

	Orbit	Perturbed
t= 100	x= 0.090272817735167835	0.090272821355768668
t= 200	x= 0.295031687482249283	0.295031884858625637
t= 300	x= 0.515226330109450181	0.515225440480693297
t= 400	x= 0.063441889347425867	0.061359558551008345

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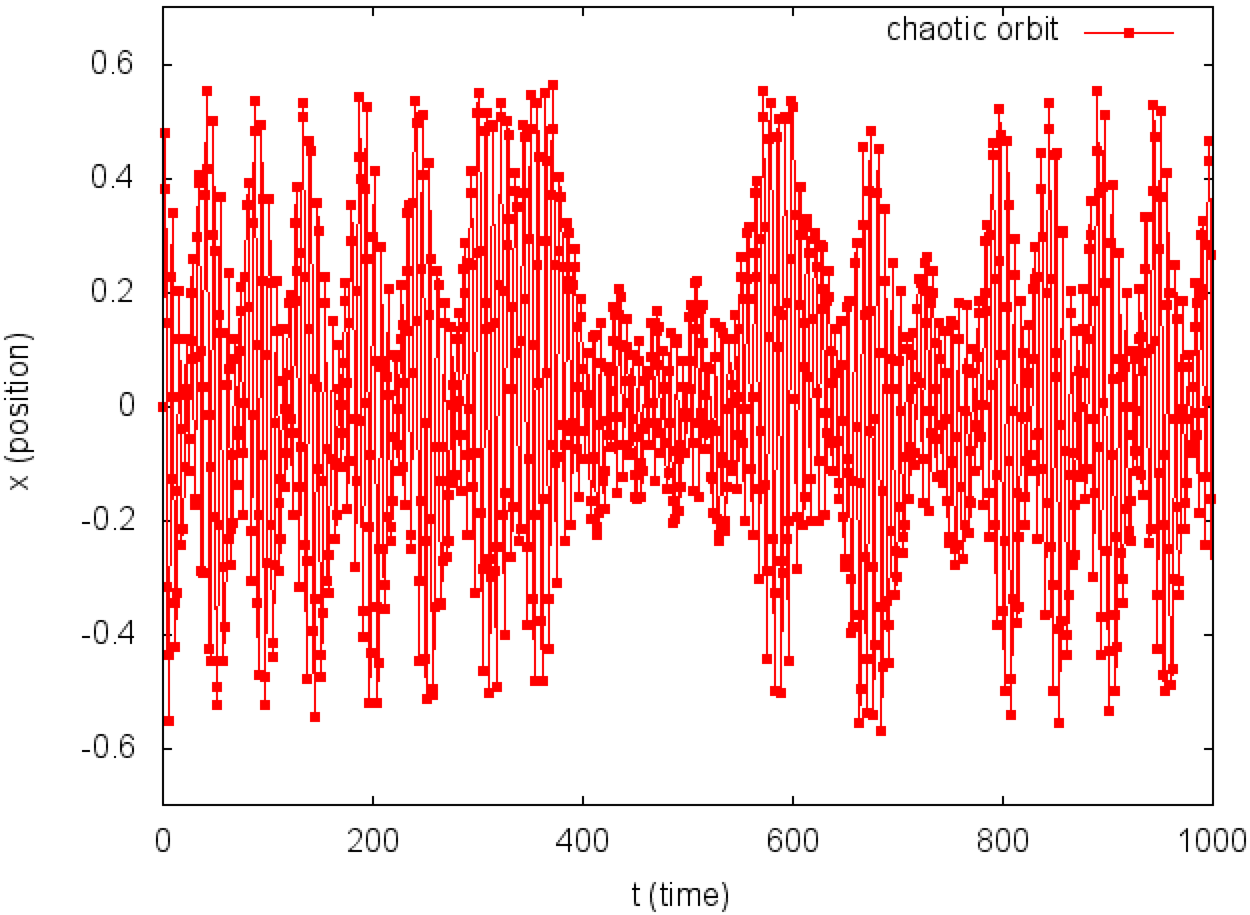
		Orbit	Perturbed
t=	100	x= 0.13299571833307644	0.13299571833727306 4
t=	5000	x= 0.37699928388910231 0	0.37699928387015657 6
t=	10000	x=-0.15909458335685522 4	-0.15909458334126030 9
t=	50000	x= 0.10199240073995576 0	0.10199240025396132 1
t=	100000	x=-0.38112053374651178 0	-0.38112053332725887 0
t=	100	x= 0.09027281773516783 5	0.09027282135576866 8
t=	200	x= 0.29503168748224928 3	0.29503188485862563 7
t=	300	x= 0.51522633010945018 1	0.51522544048069329 7
t=	400	x= 0.06344188934742586 7	0.06135955855100834 5
t=	500	x= 0.07835771929052352 8	-0.27081102267434109 5

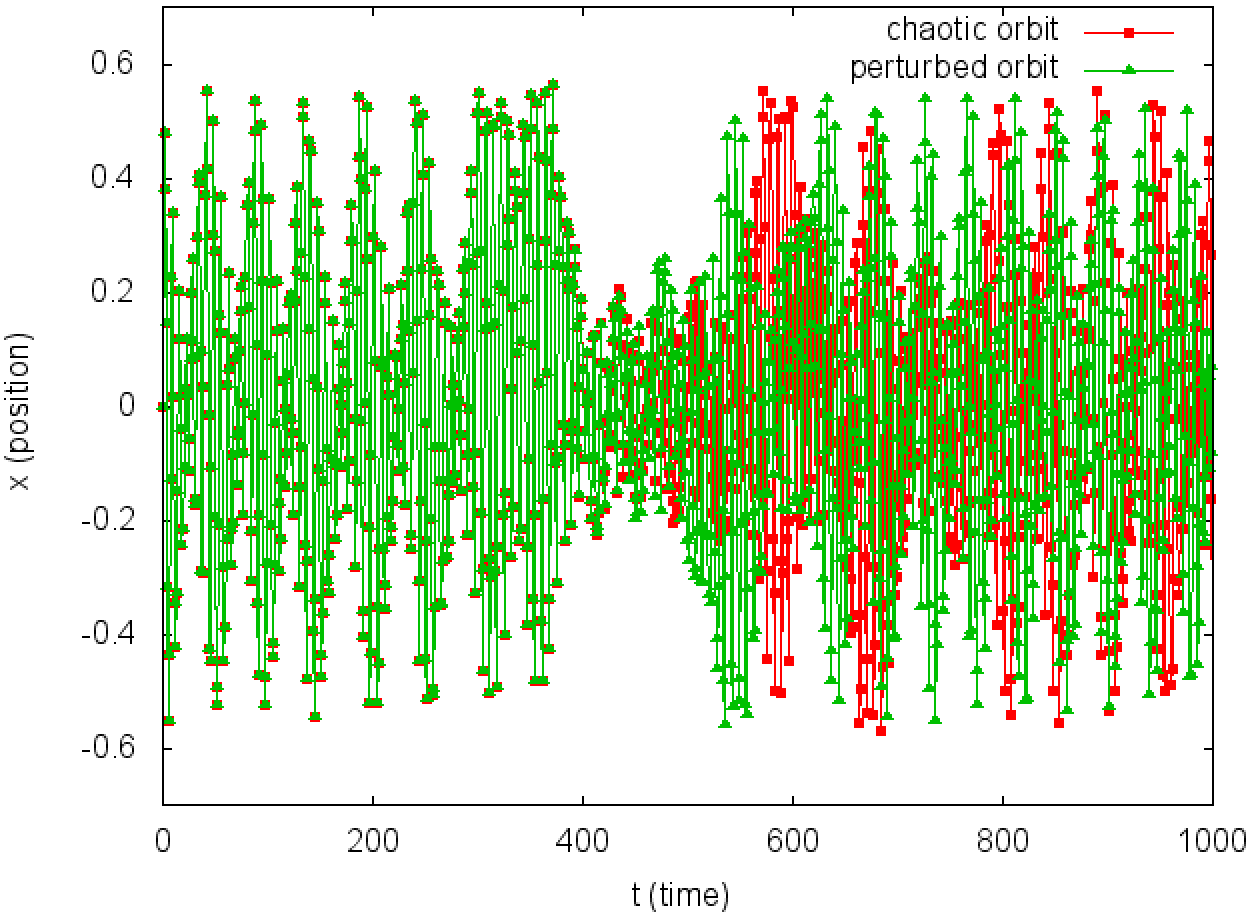
Regular vs Chaotic orbits

Chaotic orbit



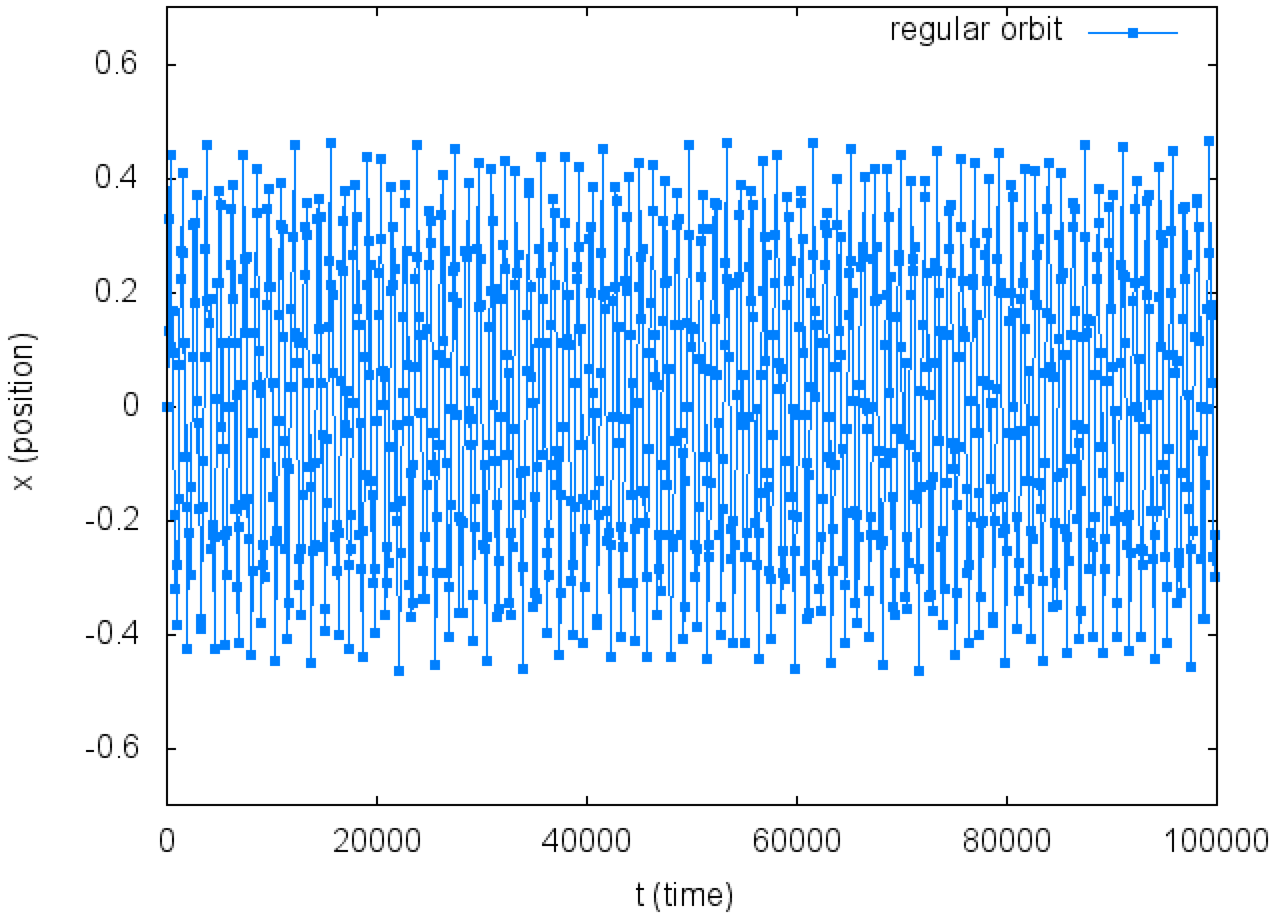
Regular vs Chaotic orbits

Chaotic orbit and its perturbation



Regular vs Chaotic orbits

Regular orbit



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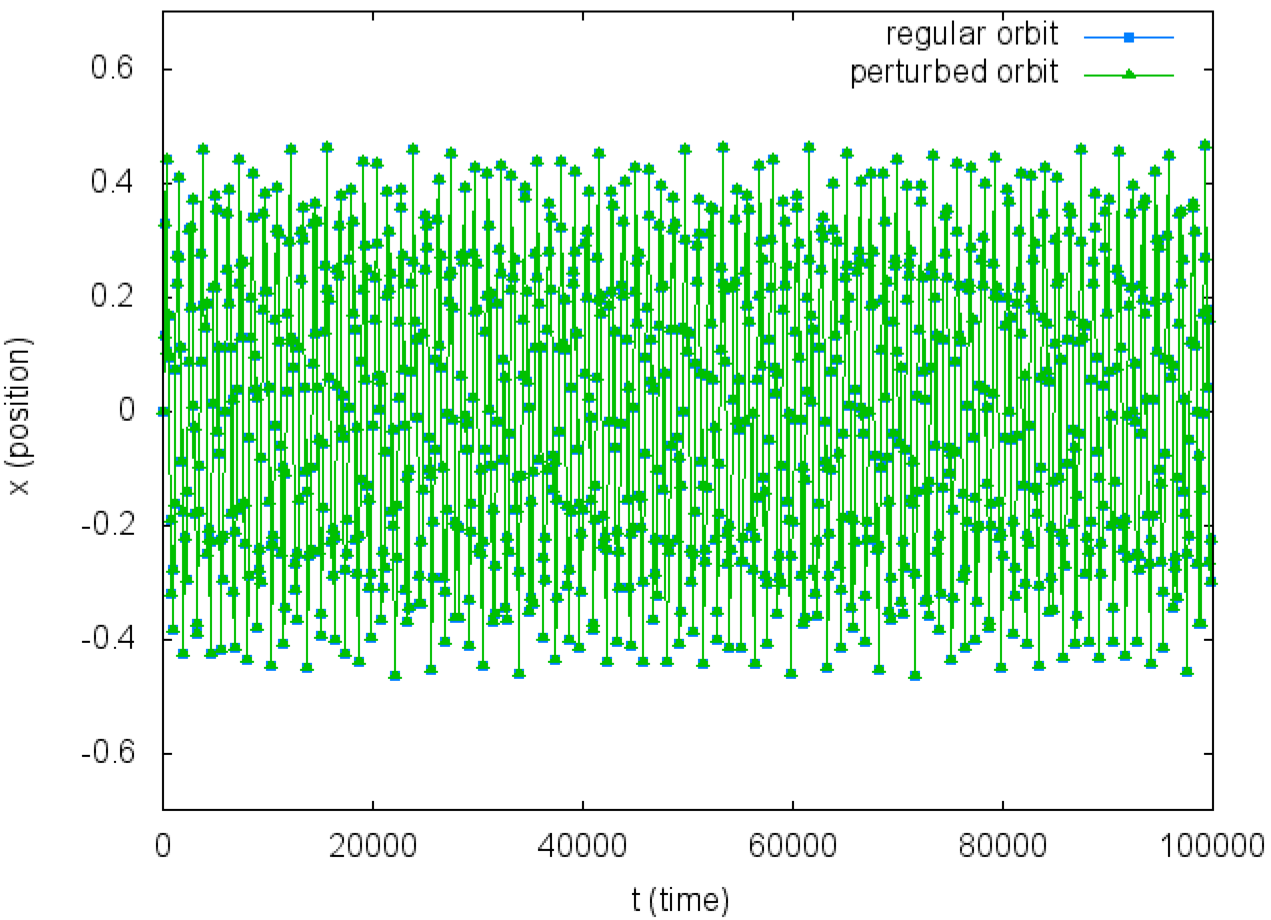

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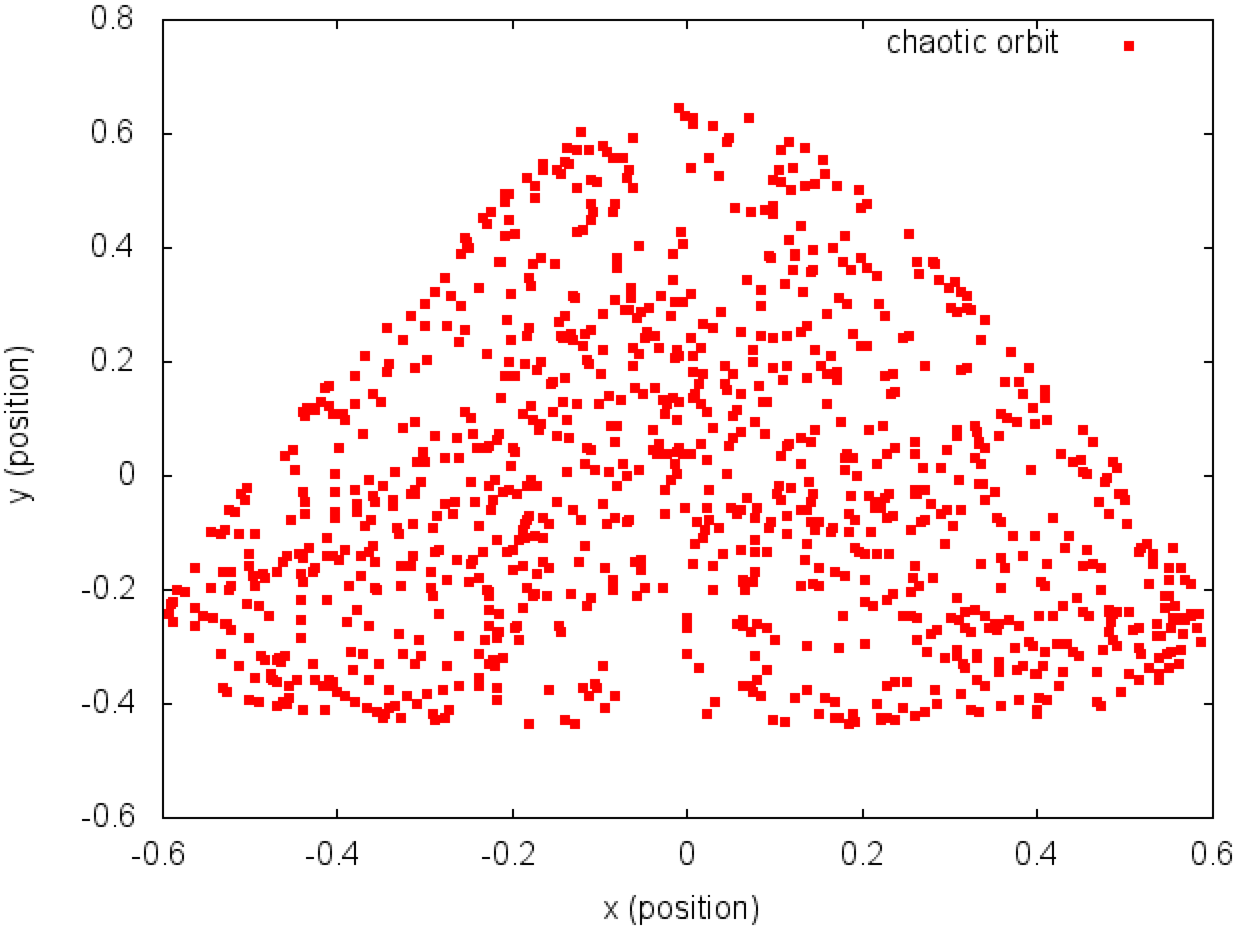
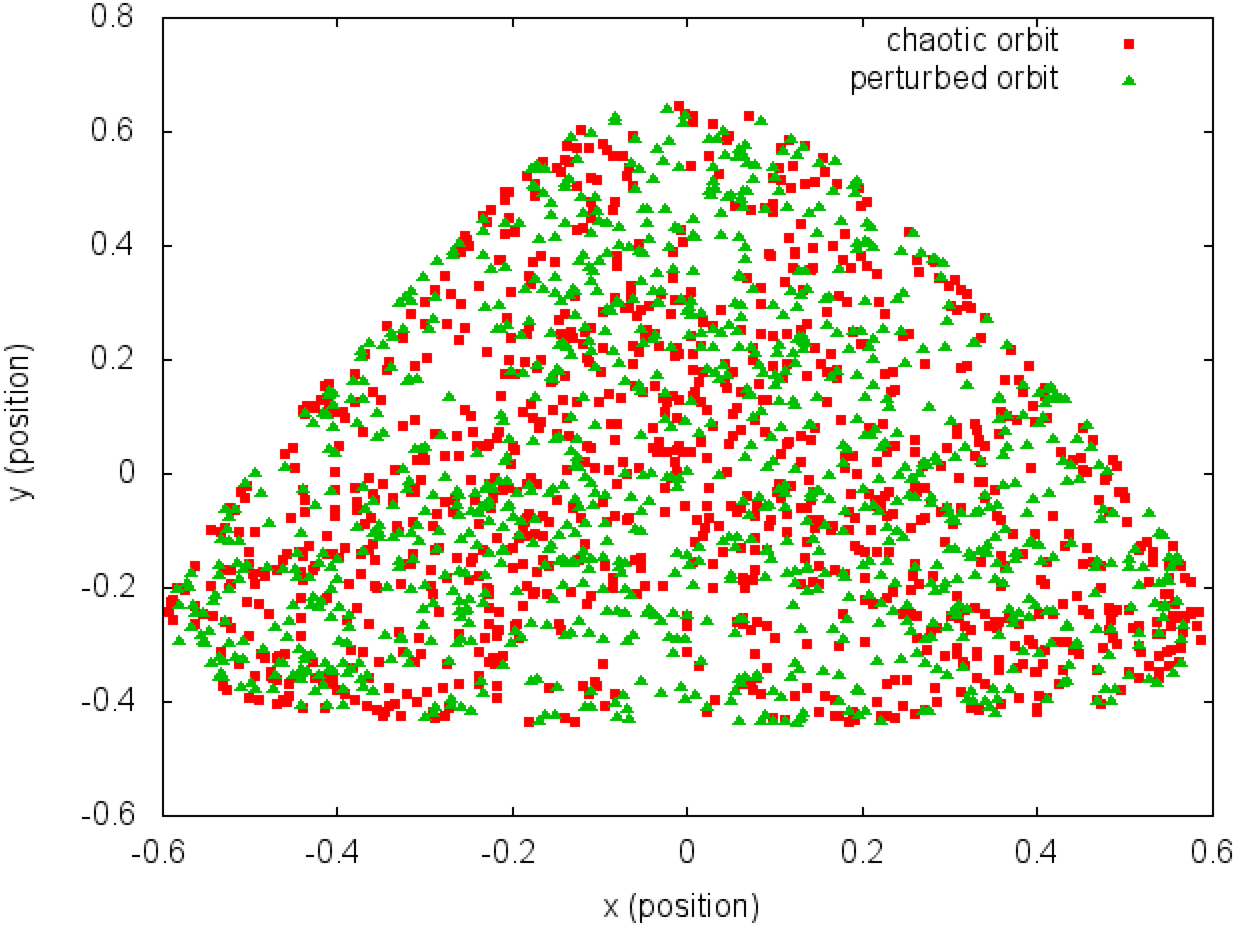
Chaotic orbit



Results for $0 \leq t \leq 10^5$

Regular vs Chaotic orbits

Chaotic orbit and its perturbation



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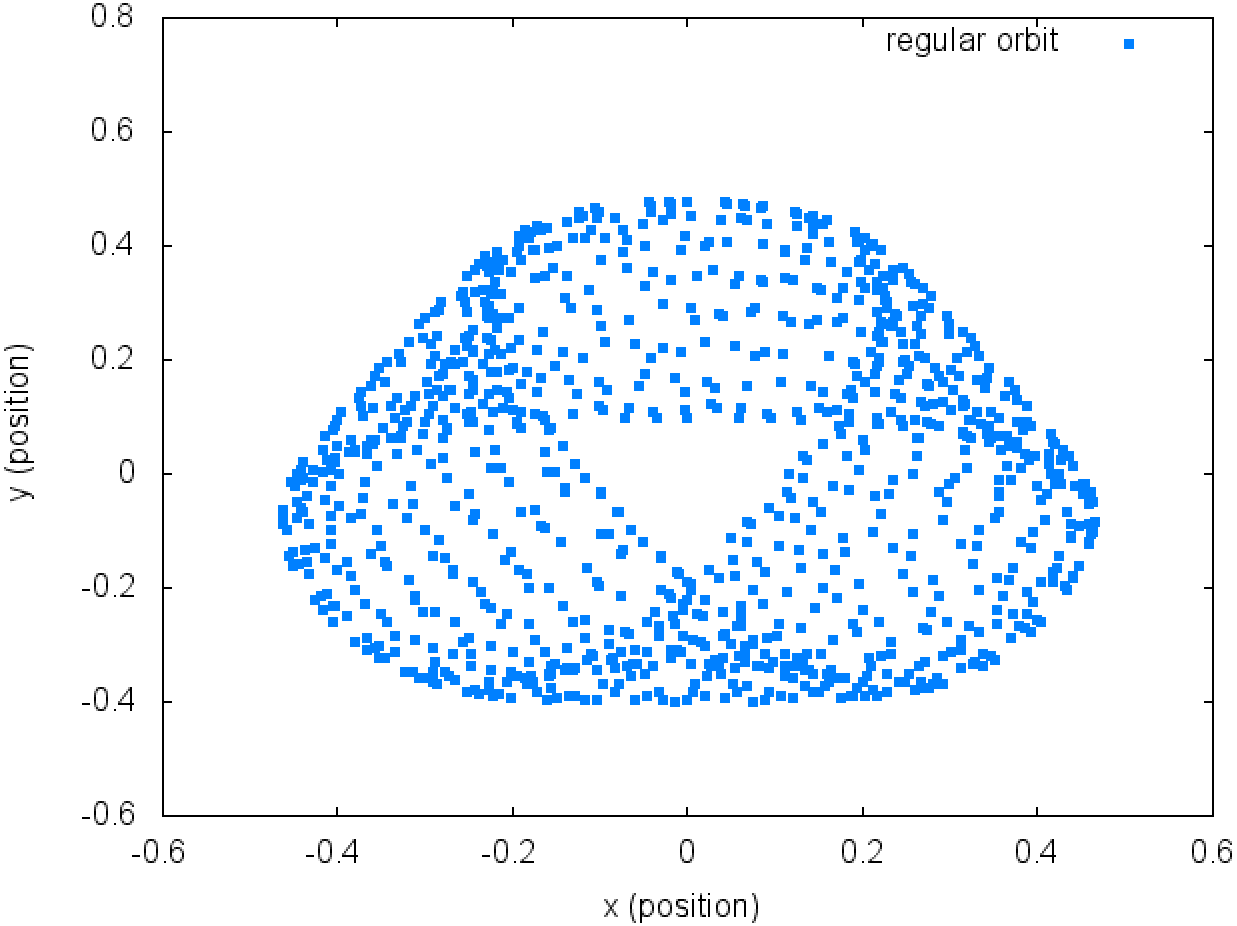
Regular vs Chaotic orbits

Regular orbit



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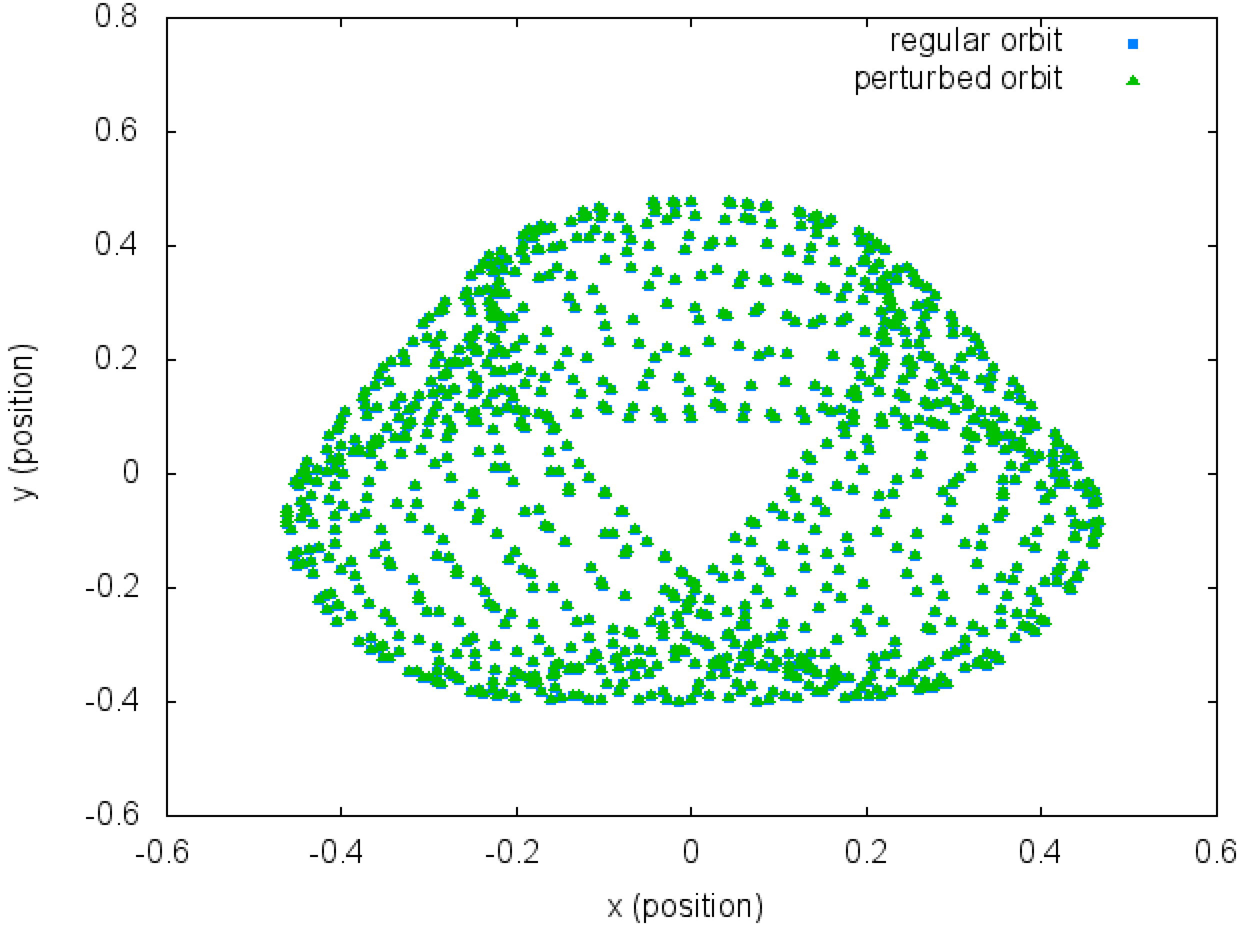

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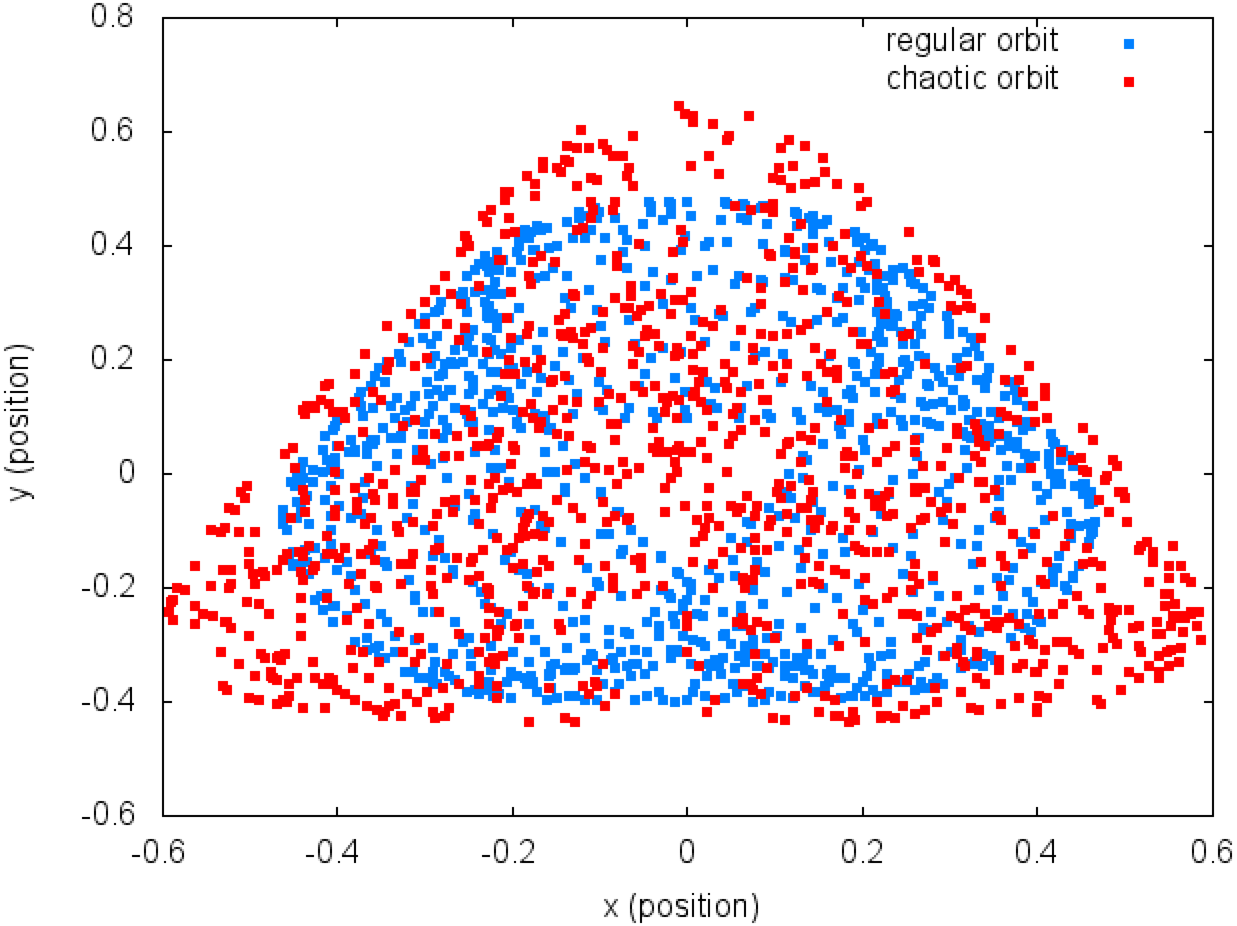
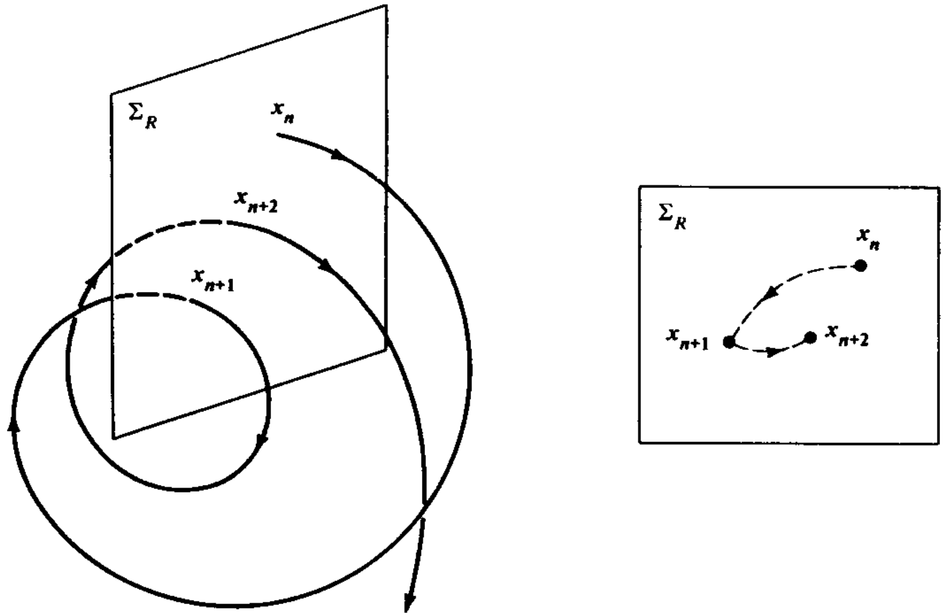


Results for $0 \leq t \leq 10^5$

Poincaré Surface of Section (PSS)

We can constrain the study of an N+1 degree of freedom Hamiltonian system to a 2N-dimensional subspace of the general phase space.



Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

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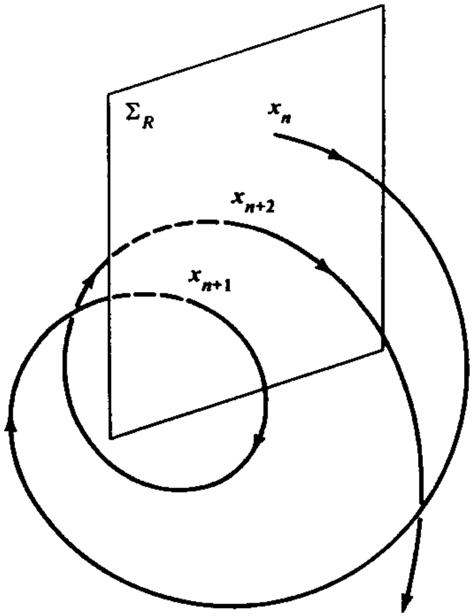

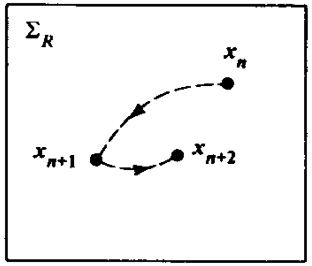

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In general we can assume a PSS of the form q_{N+1}=constant. Then only variables $q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N$ are needed to describe the evolution of an orbit on the PSS, since p_{N+1} can be found from the Hamiltonian.

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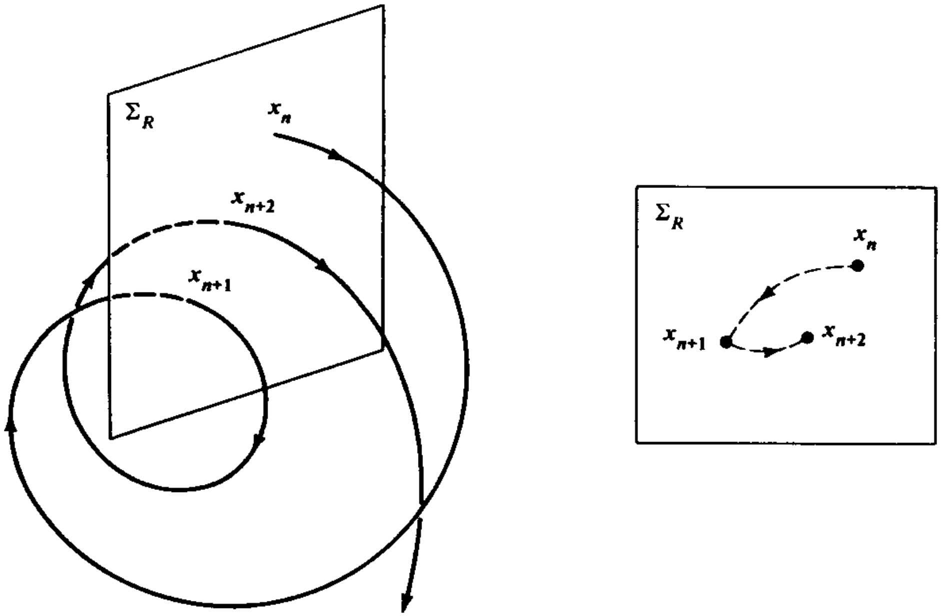

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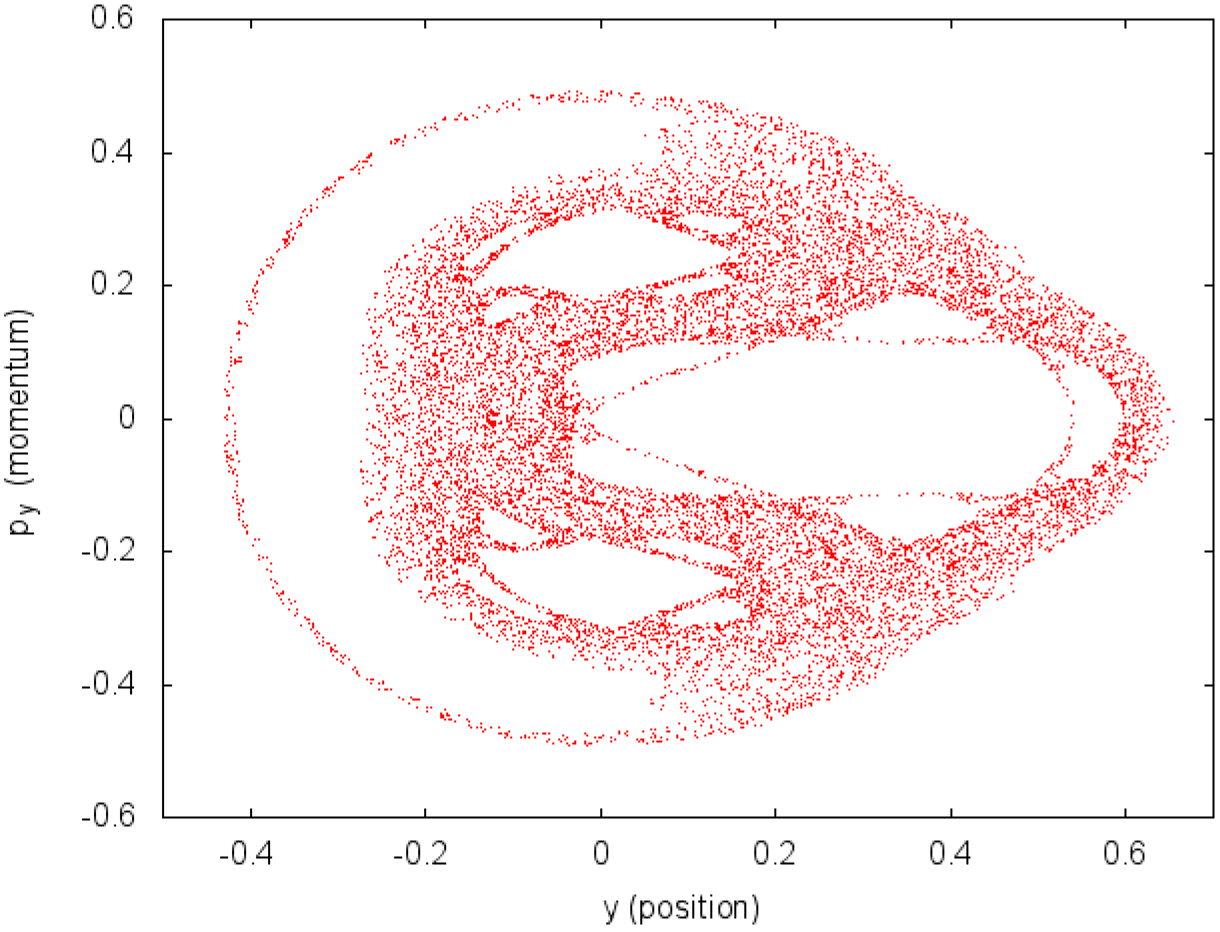
In this sense an N+1 degree of freedom Hamiltonian system corresponds to a 2N-dimensional map.

Hénon-Heiles system: PSS (x=0)



Chaotic orbit

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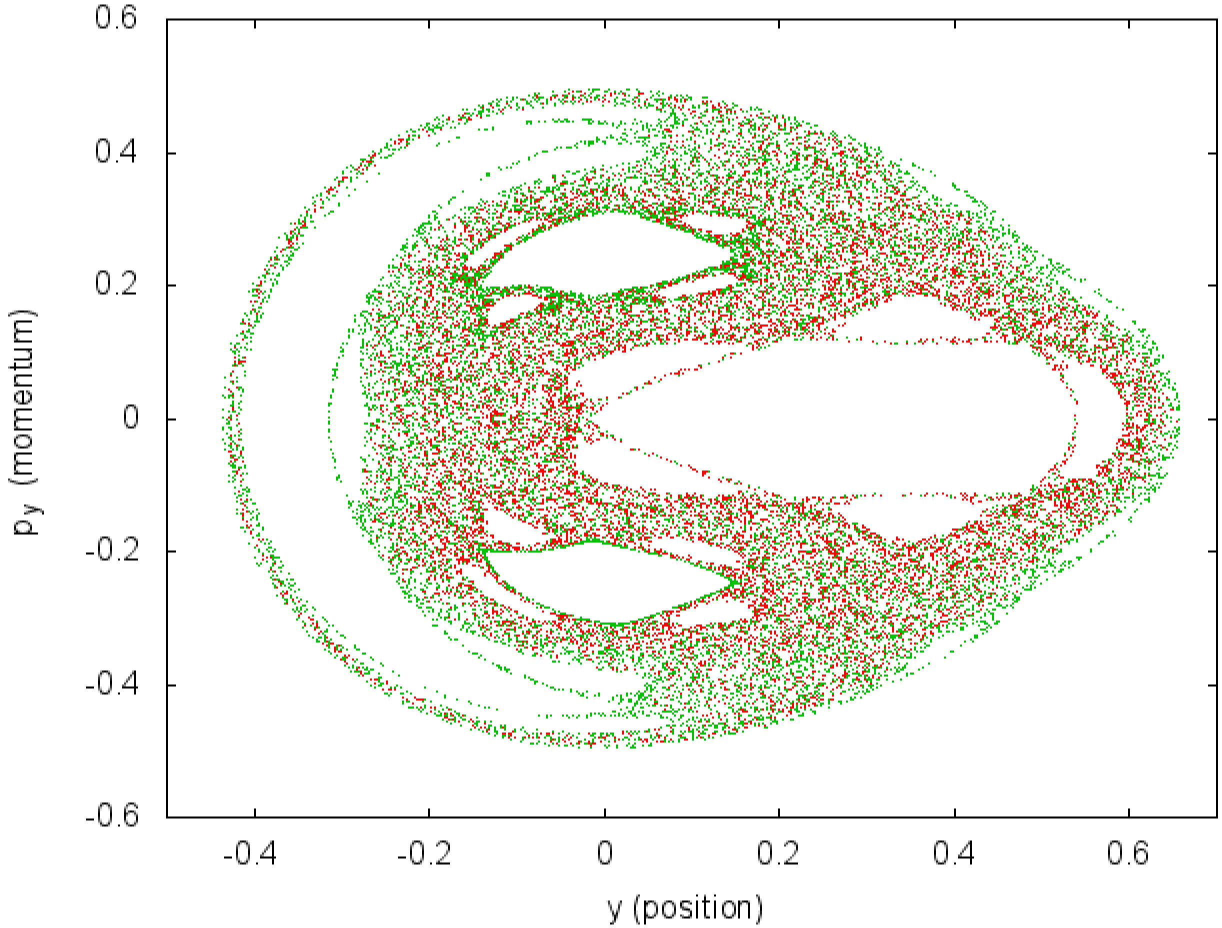

Chaotic orbit - Perturbed chaotic orbit

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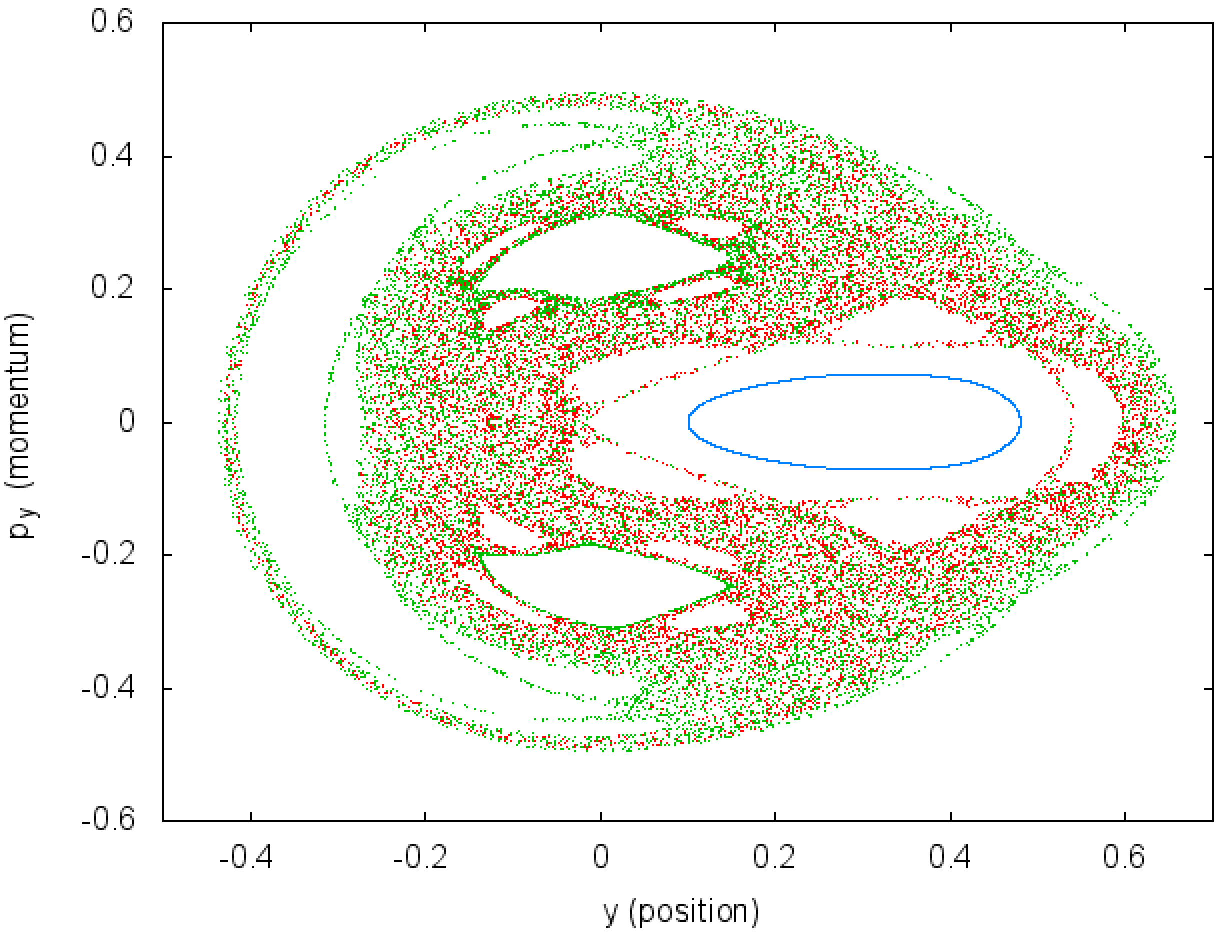

Chaotic orbit - Perturbed chaotic orbit

Regular orbit

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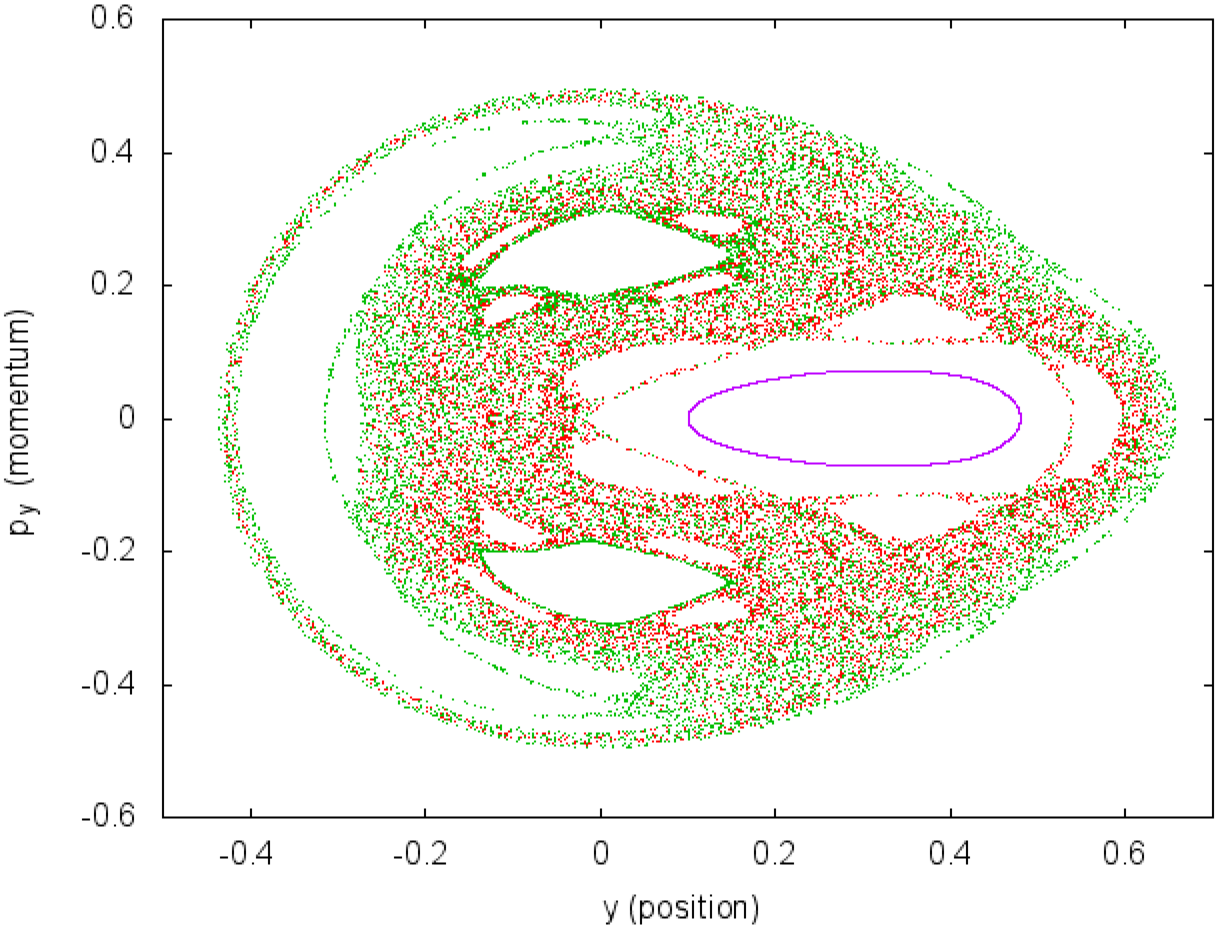
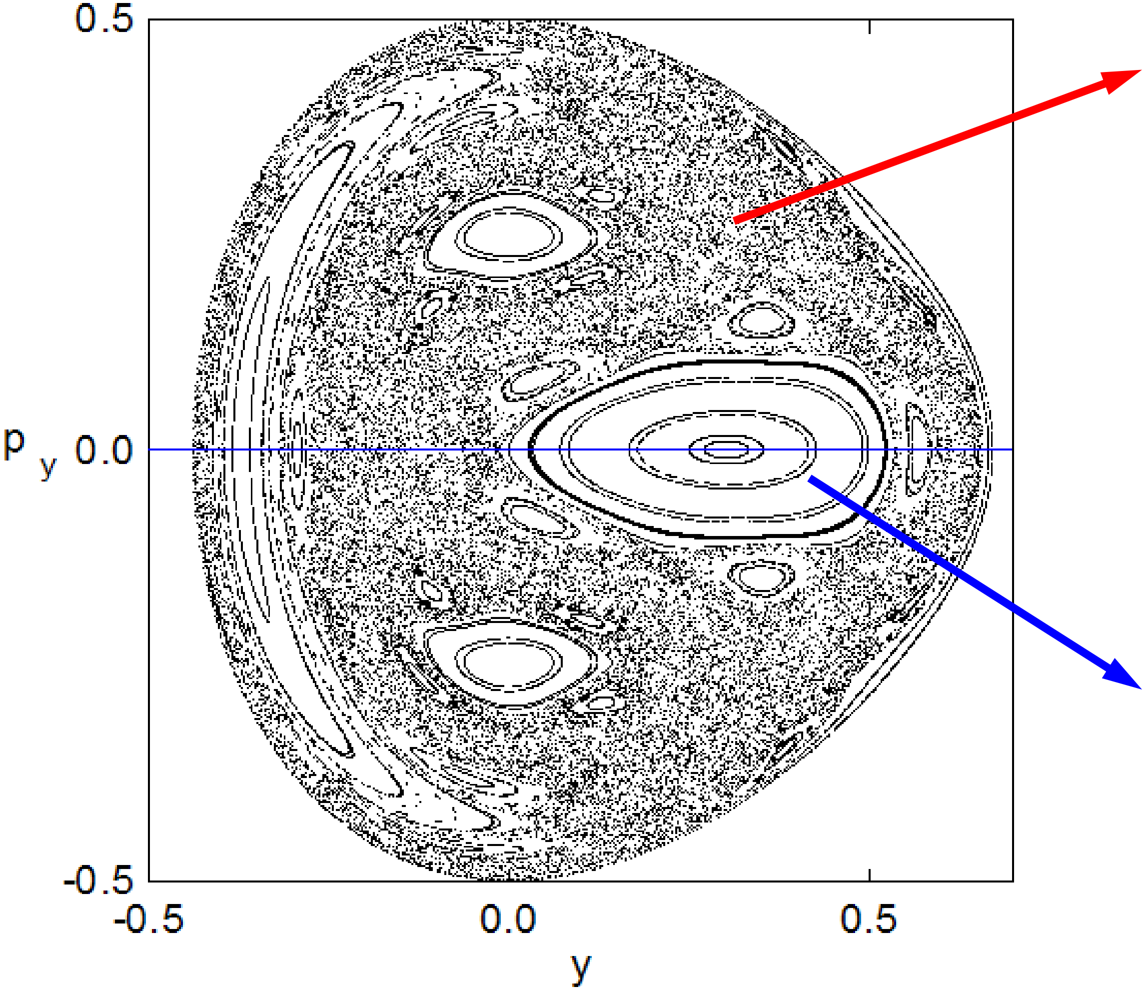


Chaotic orbit - Perturbed chaotic orbit

Regular orbit - Perturbed regular orbit

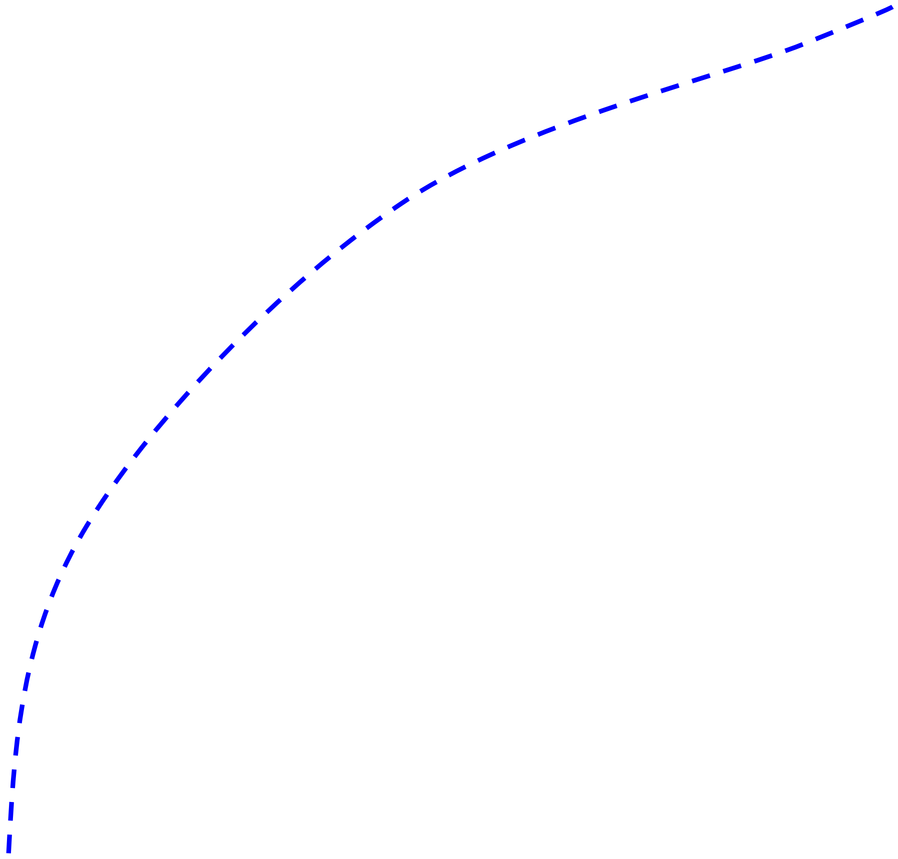
Hénon-Heiles system: PSS (x=0)



Chaotic motion
Chaotic sea

Regular motion
Island of stability

Computation of the PSS



Computation of the PSS

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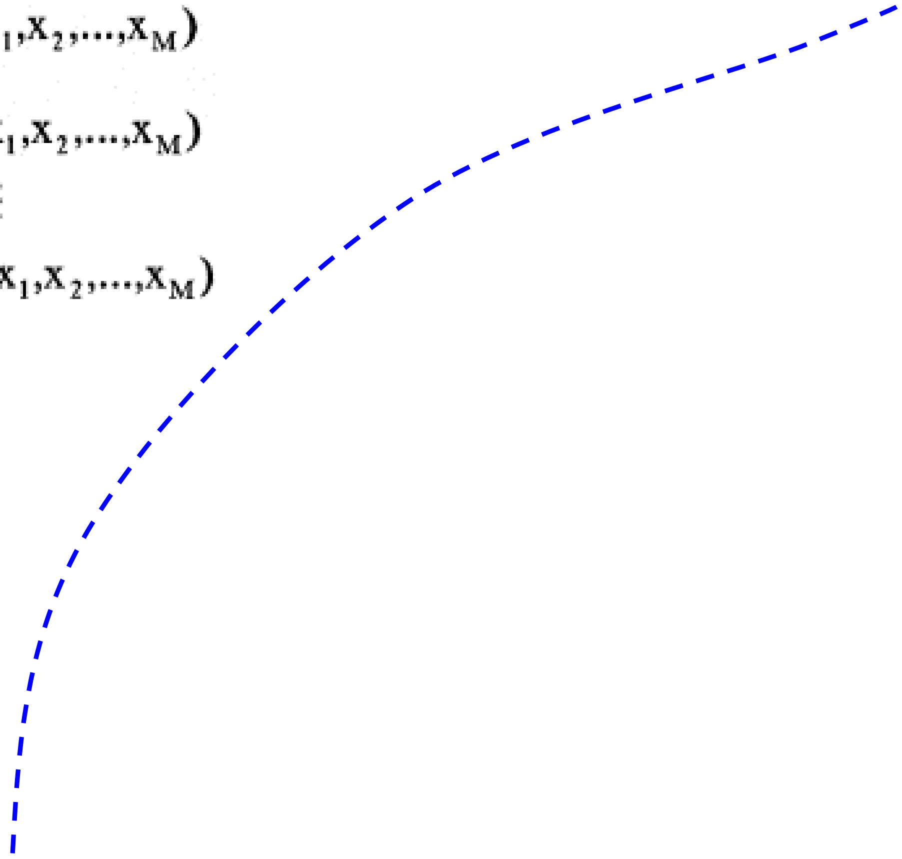

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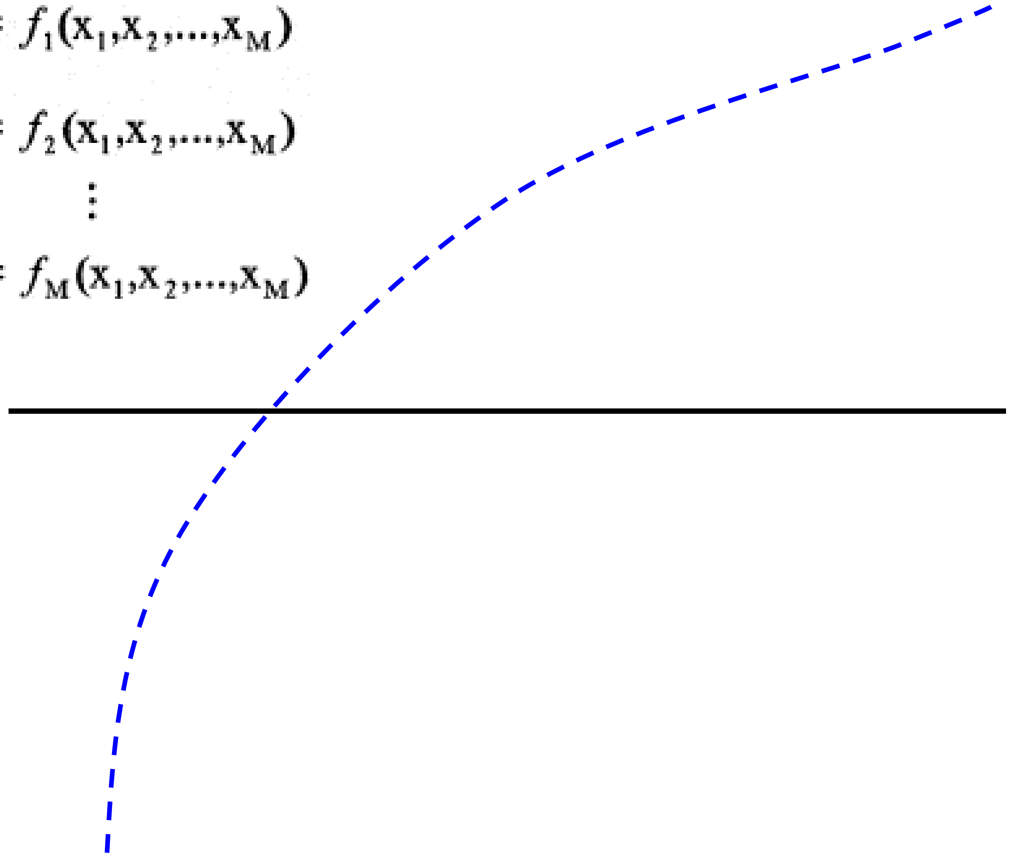

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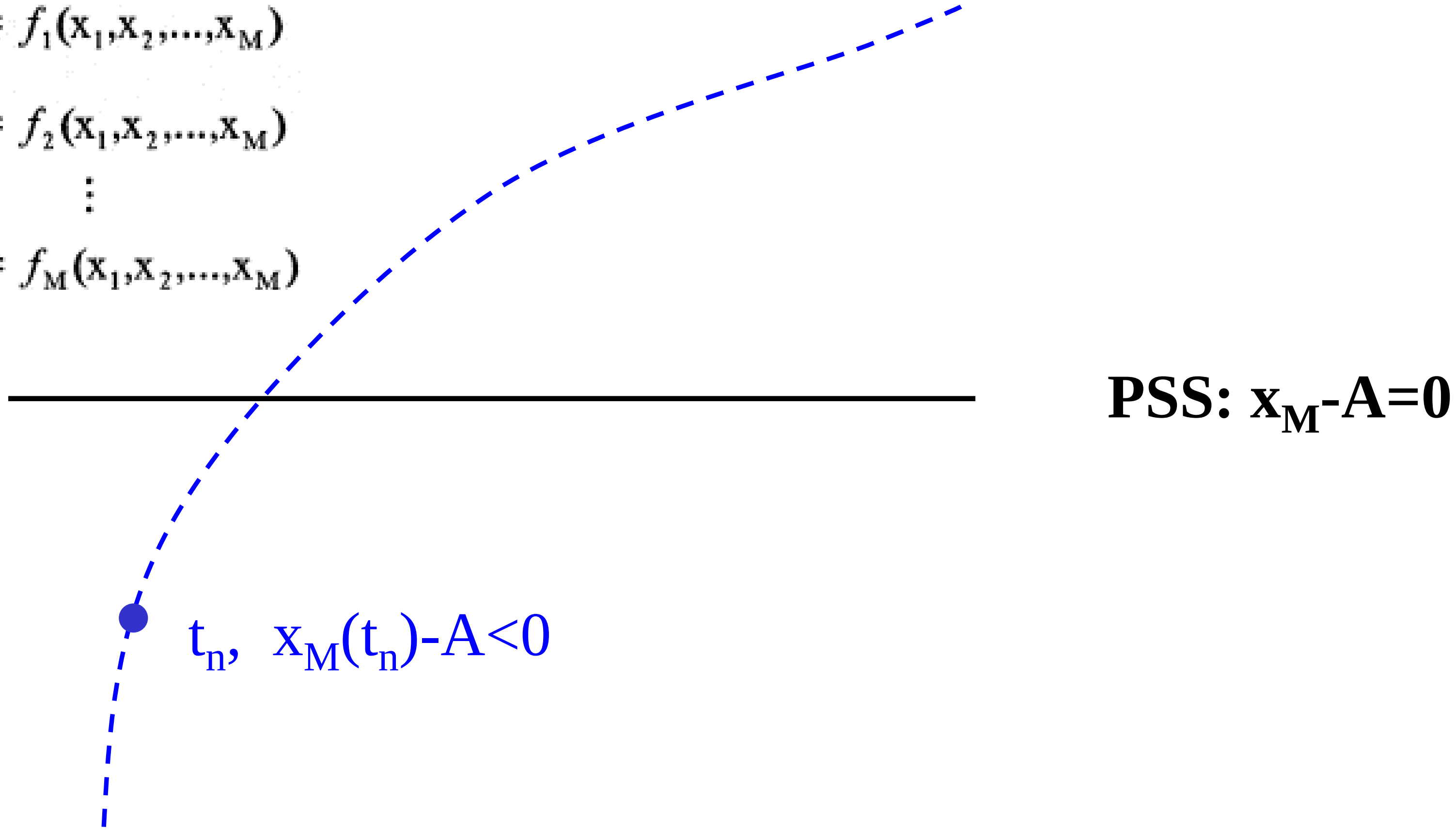

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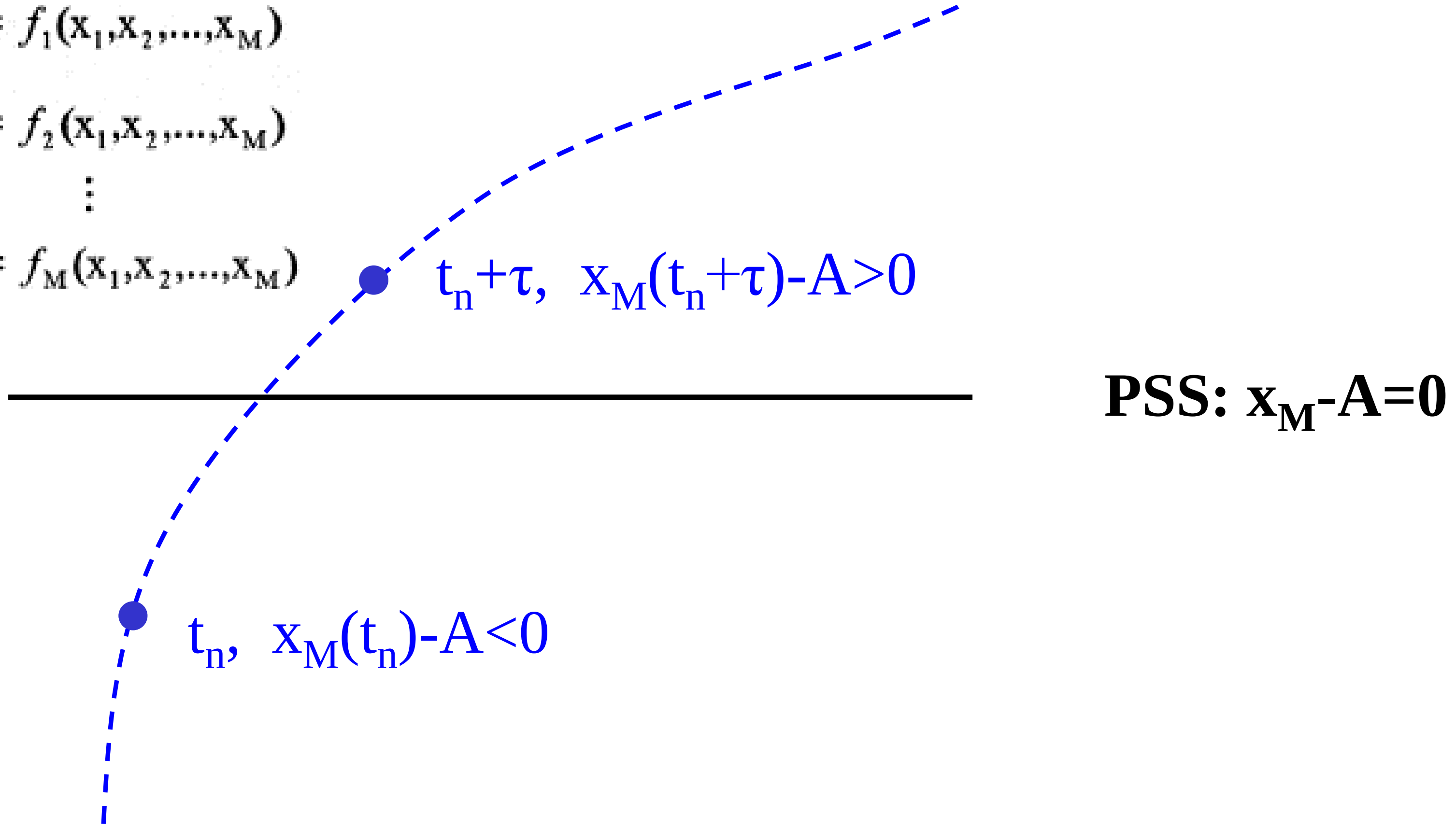

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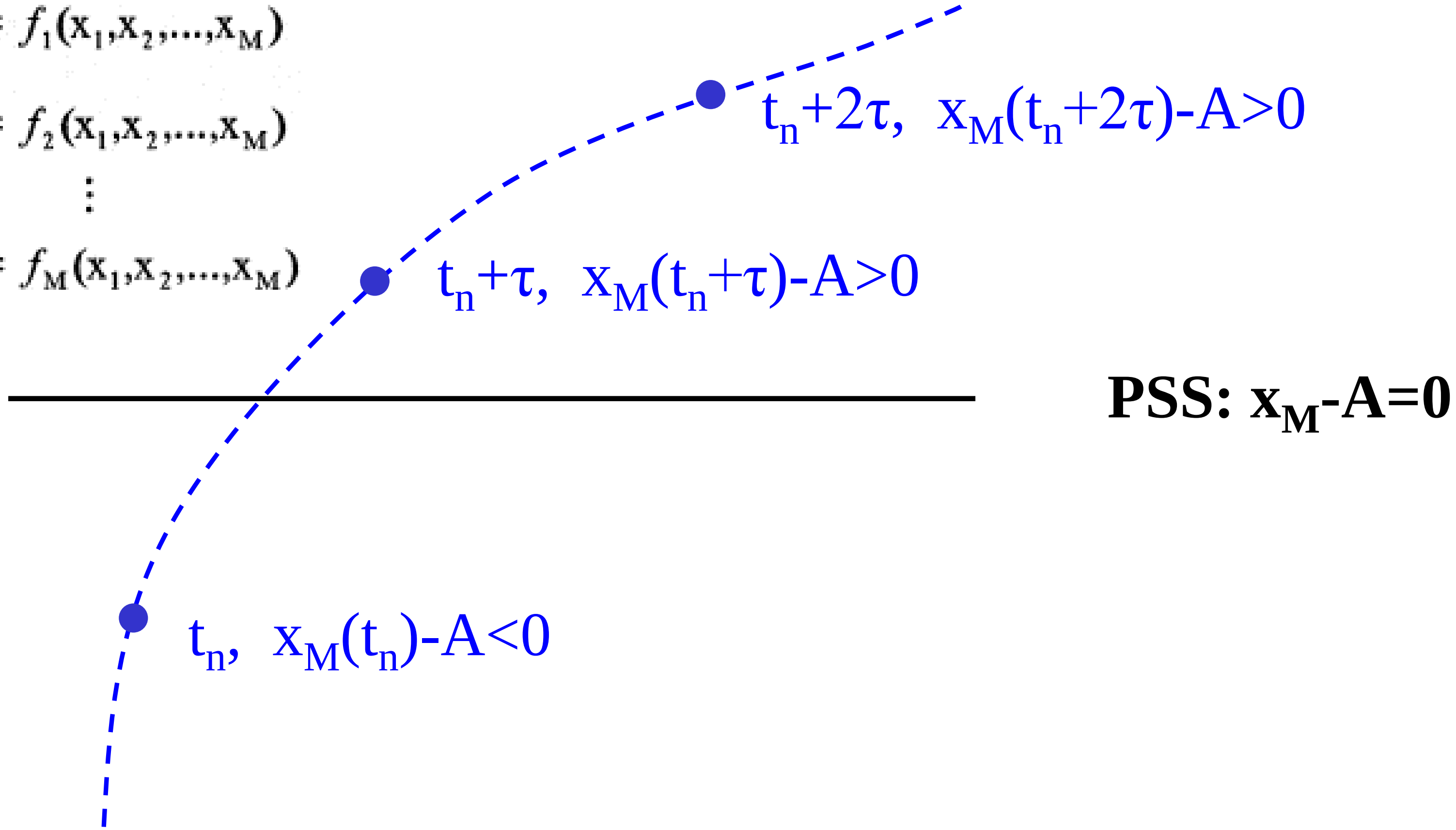

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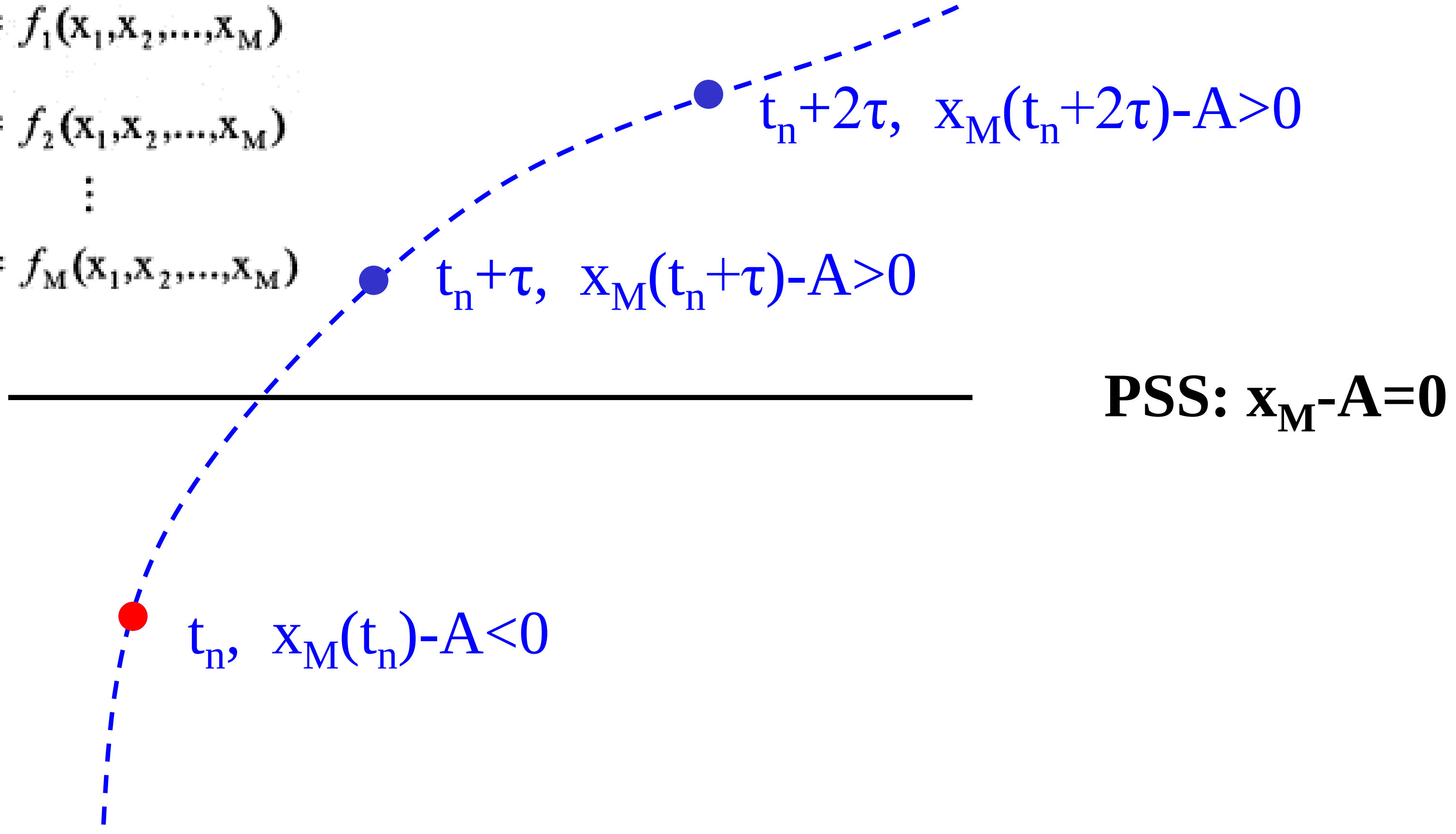

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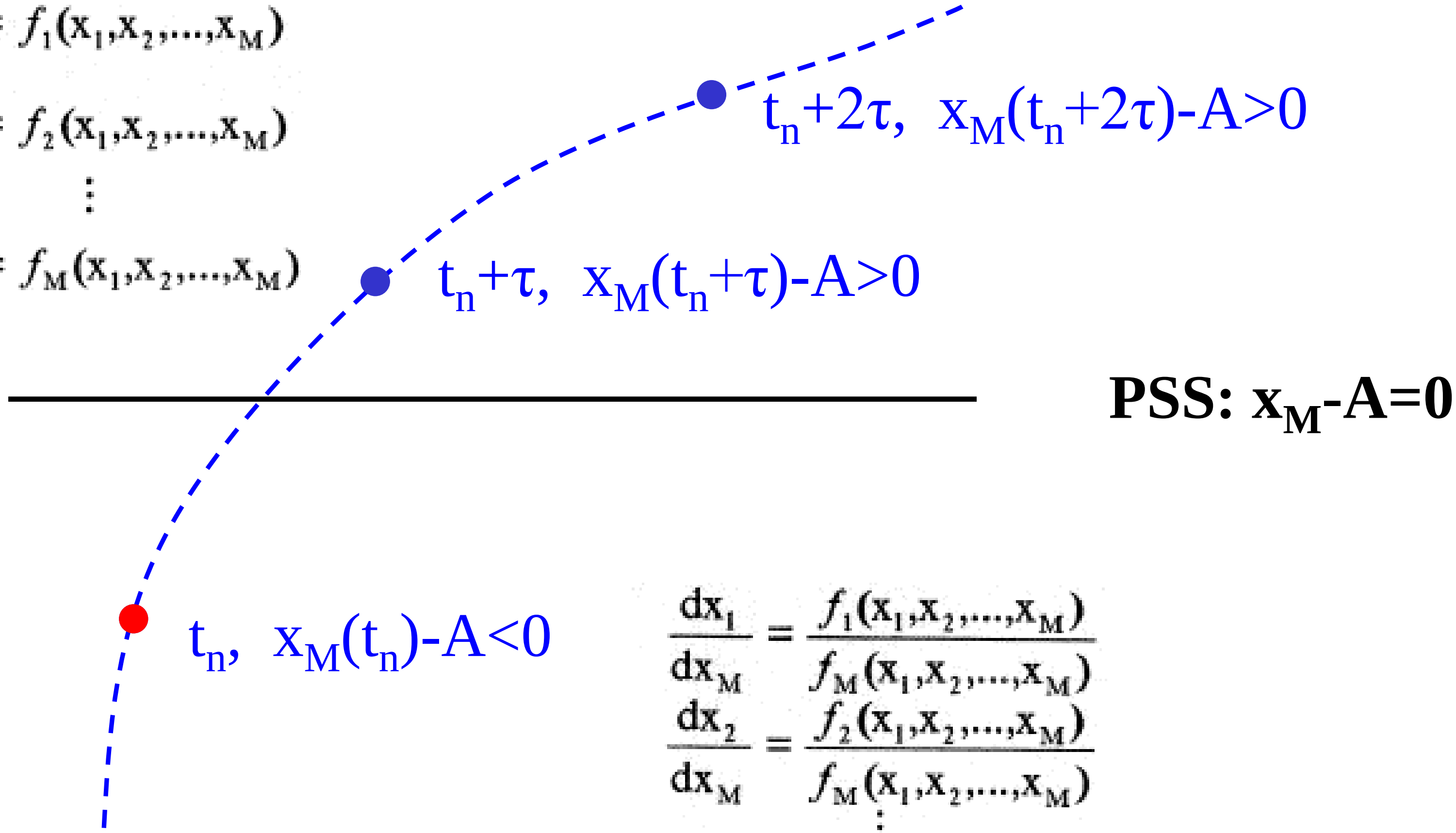

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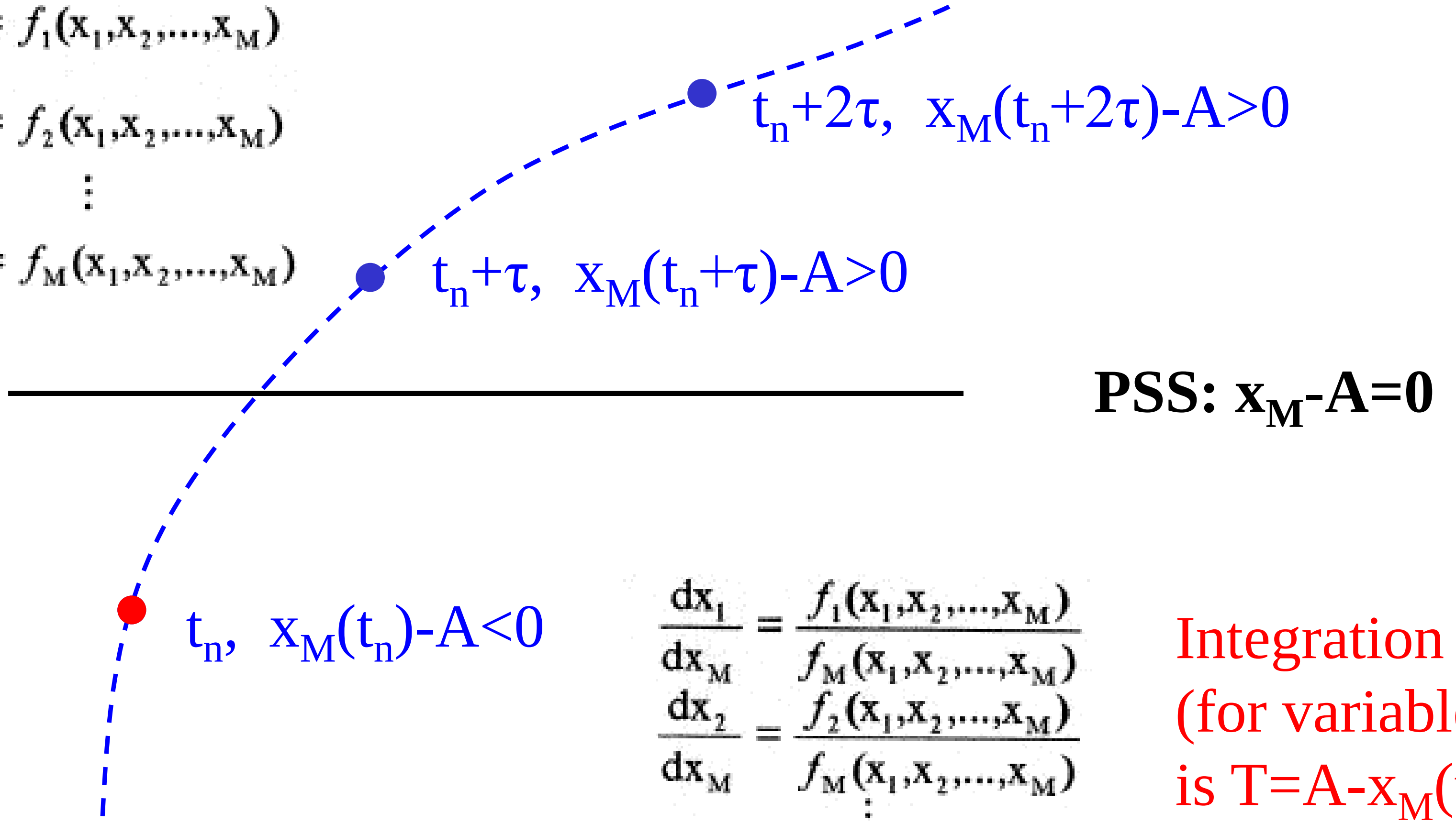

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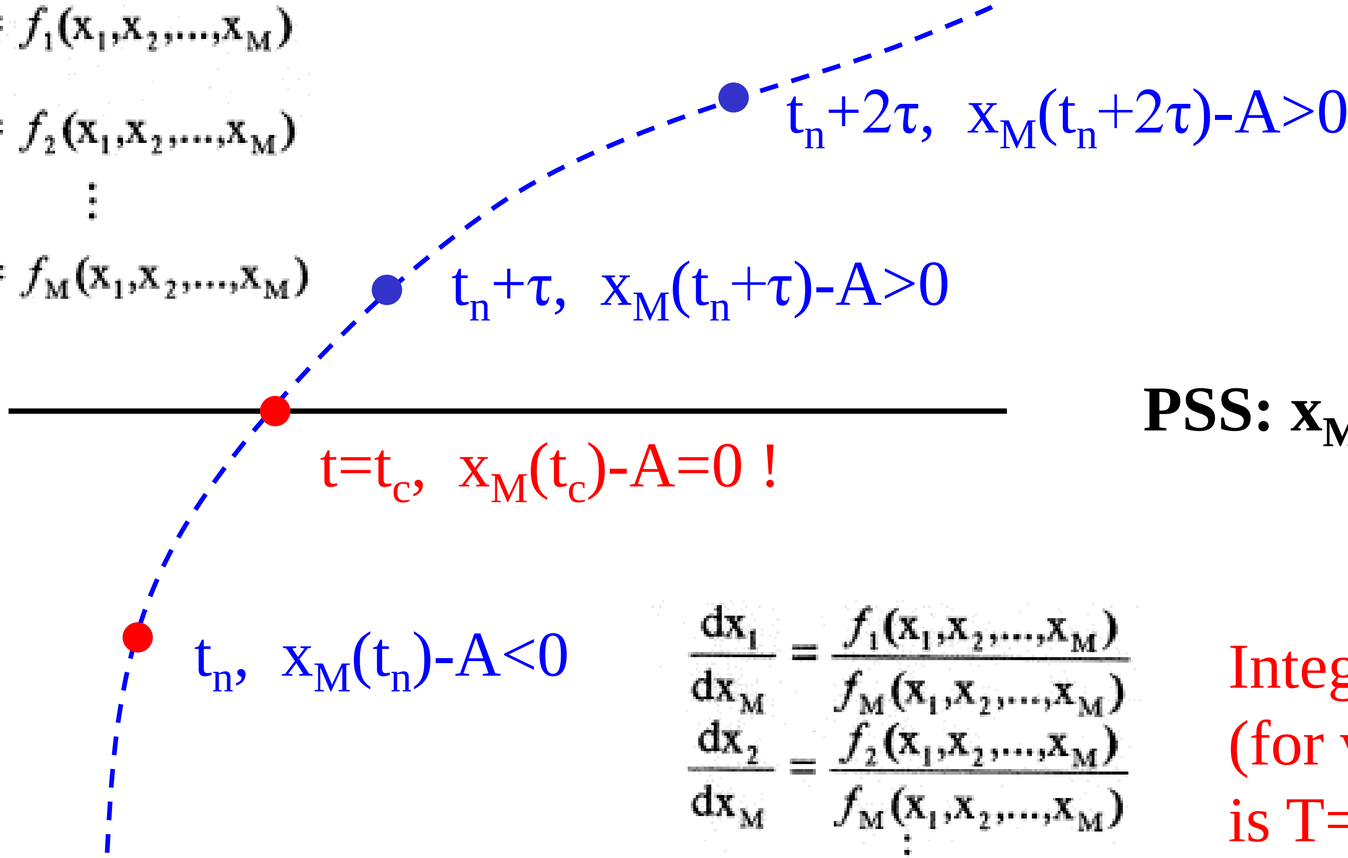

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τ=t,
K=1

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$\tau = t$, $K = 1$:

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$\tau = x_M$, $K = 1/f_M(x_1, x_2, \ldots, x_M)$:

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Chaos detection techniques

- **Based on the visualization of orbits**
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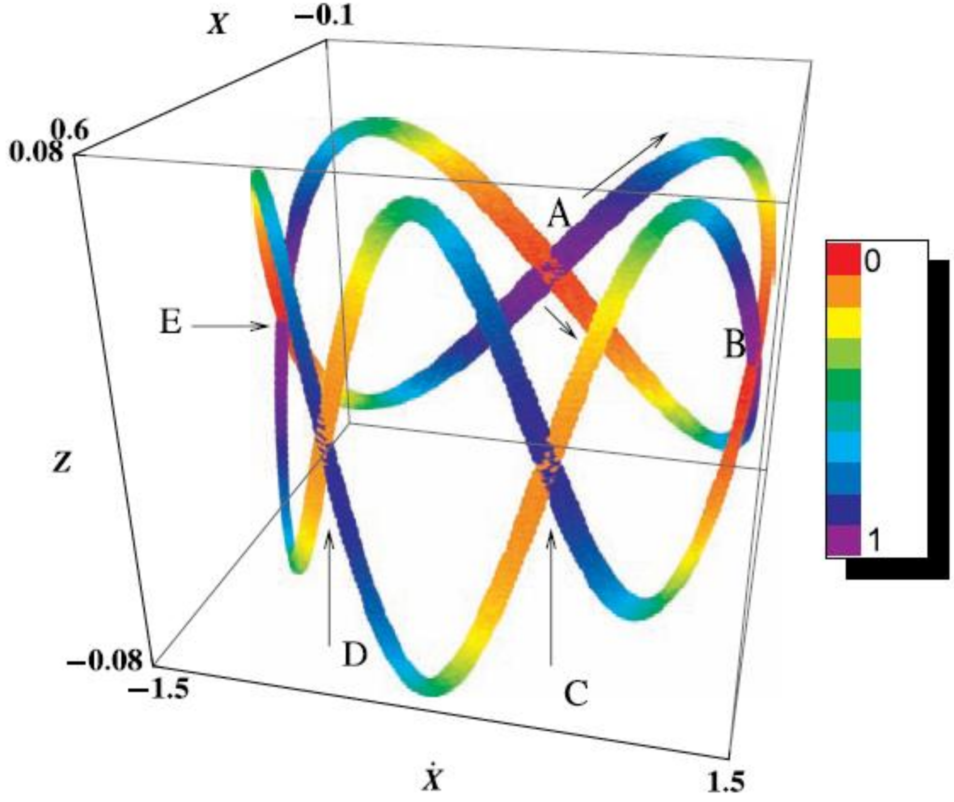
  - ✓ **the color and rotation (CR) method**
  - ✓ **the 3D phase space slices (3PSS) technique**

The color and rotation (CR) method

For 3 degree of freedom Hamiltonian systems and 4 dimensional symplectic maps:

We consider the 3D projection of the PSS and use color to indicate the 4th dimension.

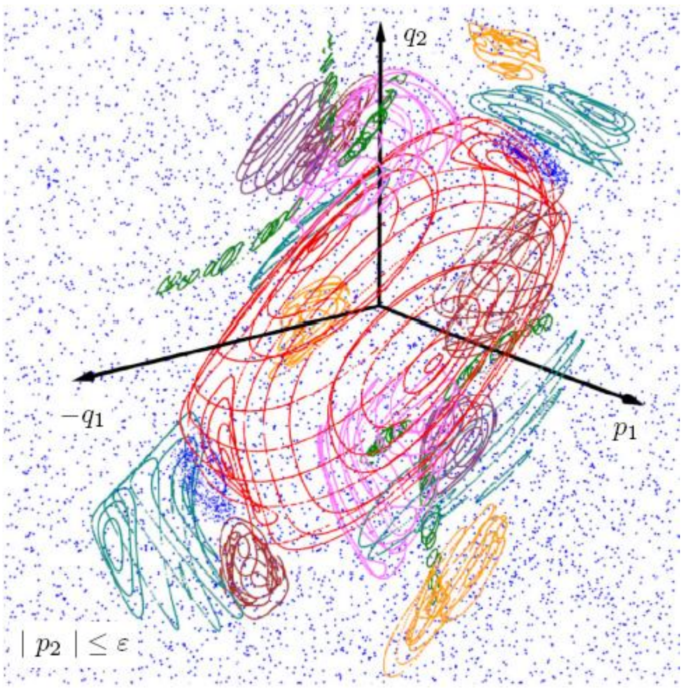


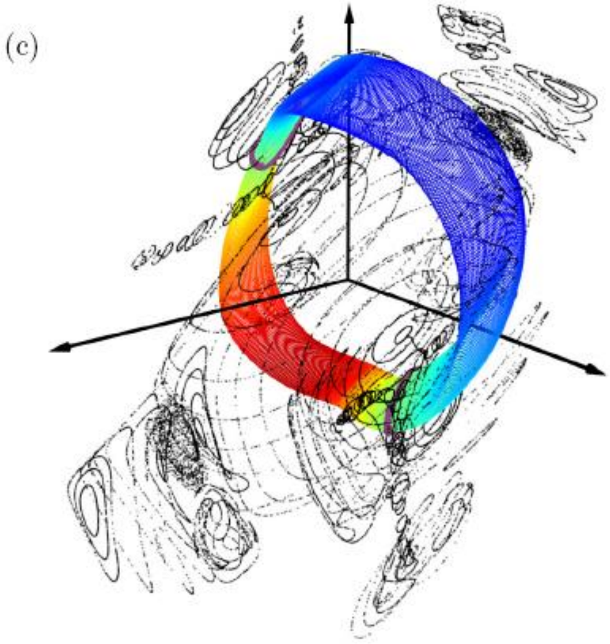
Katsanikas M and Patsis P A 2011 Int. J. Bif. Chaos 21 467

The 3D phase space slices (3PSS) technique

For 3 degree of freedom Hamiltonian systems and 4 dimensional symplectic maps:

We consider thin 3D phase space slices of the 4D phase space (e.g. $|p_2| \leq \varepsilon$) and present intersections of orbits with these slices.





Richter et al. 2014 Phys. Rev. E 89 022902

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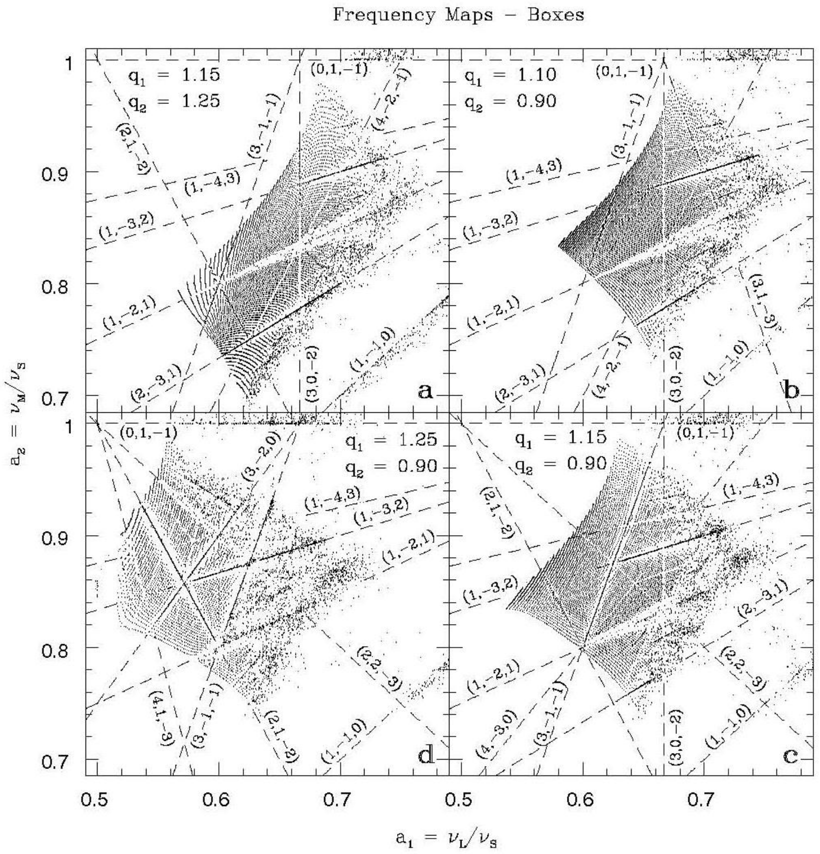
- **Based on the numerical analysis of orbits**
  - ✓ **Frequency Map Analysis**
  - ✓ **0-1 test**

Frequency Map Analysis

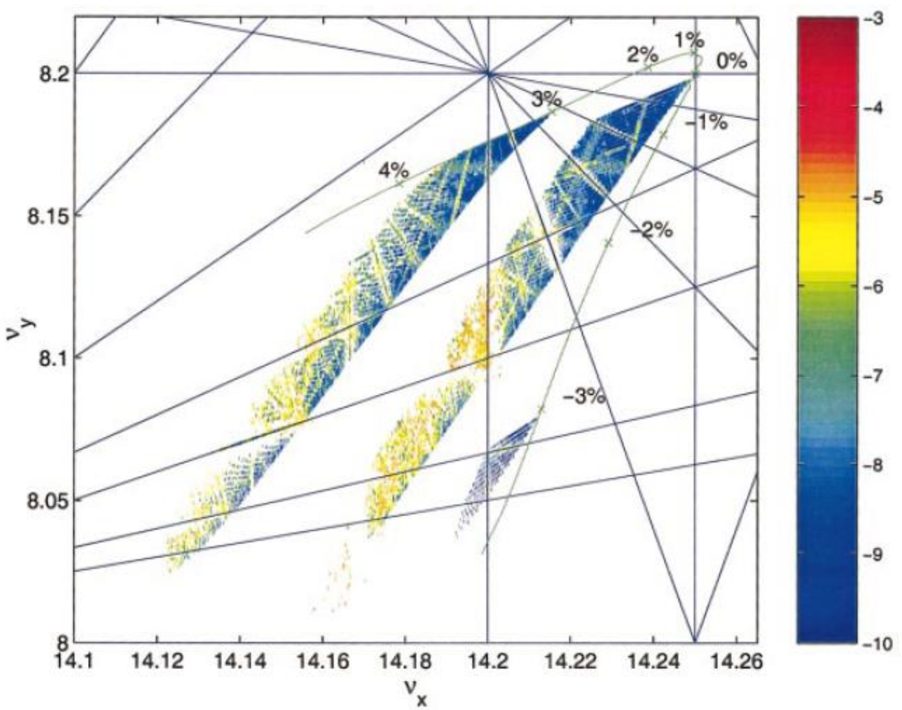
Create Frequency Maps by computing the fundamental frequencies of orbits.

Regular motion: The computed frequencies do not vary in time

Chaotic motion: The computed frequencies vary in time



Papaphilippou Y and Laskar J 1998 Astron. Astrophys. 329 451



Steier C et al. 2002 Phys. Rev. E 65 056506

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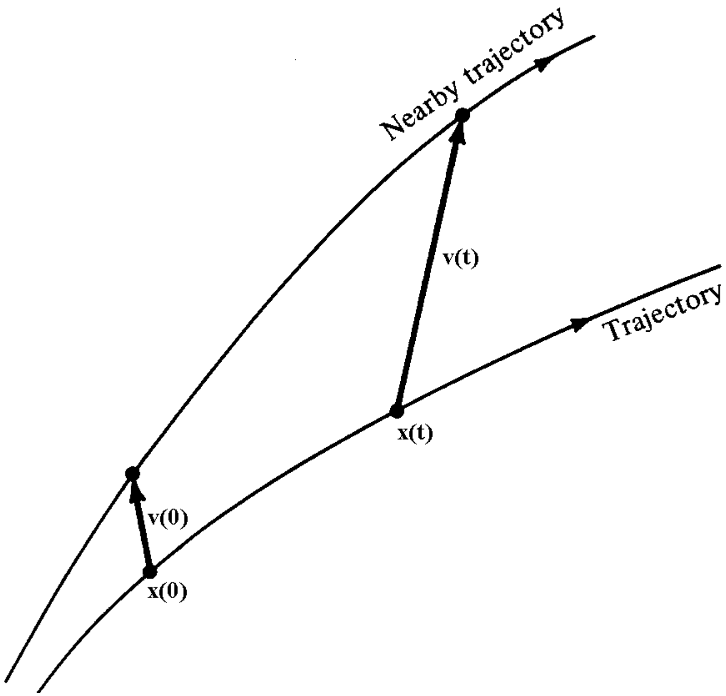
- **Chaos indicators based on the evolution of deviation vectors from a given orbit**
  - ✓ **Maximum Lyapunov Exponent**
  - ✓ **Fast Lyapunov Indicator (FLI) and Orthogonal Fast Lyapunov Indicators (OFLI and OFLI2)**
  - ✓ **Mean Exponential Growth Factor of Nearby Orbits (MEGNO)**
  - ✓ **Relative Lyapunov Indicator (RLI)**
  - ✓ **Smaller ALignment Index – SALI**
  - ✓ **Generalized ALignment Index – GALI**

Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2,\ldots,\delta x_n)^T \text{, with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i, j = 1, 2, \ldots, n$$

Example (Hénon-Heiles system)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

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In order to get the variational equations we linearize the above equations by substituting x, y, px, py with $x+v_1$, $y+v_2$, p_x+v_3, p_y+v_4 where $v=(v_1,v_2,v_3,v_4)$ is the deviation vector. So we get:

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

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$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \boxed{\dot{p}_x = -x - 2xy} \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

In order to get the variational equations we linearize the above equations by substituting x, y, px, py with $x+v_1$, $y+v_2$, p_x+v_3, p_y+v_4 where $v=(v_1,v_2,v_3,v_4)$ is the deviation vector. So we get:

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2xy - 2xv_2 - 2yv_1 - 2v_1v_2 \Rightarrow$$

$$\dot{v}_3 = -v_1 - 2yv_1 - 2xv_2$$

Example (Hénon-Heiles system)

Variational equations:

$$\frac{dv}{dt} = -J \cdot P \cdot v$$

Example (Hénon-Heiles system)

Variational equations:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

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Example (Hénon-Heiles system)

Variational equations: $\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$

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Example (Hénon-Heiles system)

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Example (Hénon-Heiles system)

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$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \dot{v}_3 &= -v_1 - 2xv_2 - 2yv_1 \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned}$$

Example (Hénon-Heiles system)

Variational equations: $\frac{dv}{dt} = -J \cdot P \cdot v$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\dot{v}_1 = v_3$$

$$\dot{v}_2 = v_4$$

$$\boxed{\dot{v}_3 = -v_1 - 2xv_2 - 2yv_1}$$

$$\dot{v}_4 = -v_2 - 2xv_1 + 2yv_2$$

Example (Hénon-Heiles system)

Variational equations: $\frac{dv}{dt} = -J \cdot P \cdot v$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \dot{v}_3 &= -v_1 - 2xv_2 - 2yv_1 \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned} \quad + \quad \begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= -y - x^2 + y^2 \end{aligned}$$

Complete set of equations

Symplectic Maps

Consider an 2N-dimensional symplectic map T. In this case we have discrete time.

This is an area-preserving map whose Jacobian matrix

$$\mathbf{M} = \frac{\partial \mathbf{T}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_{2N}} \\ \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_{2N}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_{2N}} \end{bmatrix}$$

satisfies

$$M^T \cdot J_{2N} \cdot M = J_{2N}$$

Symplectic Maps

Consider an 2N-dimensional symplectic map T. In this case we have discrete time.

The evolution of an orbit with initial condition

$$P(0)=(x_1(0), x_2(0),\dots,x_{2N}(0))$$

is governed by the equations of map T

$$P(i+1)=T\,P(i)\;\;,\;\; i=0,1,2,\dots$$

The evolution of an initial deviation vector

$$v(0) = (\delta x_1(0), \delta x_2(0),\dots, \delta x_{2N}(0))$$

is given by the corresponding tangent map

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)\;\;, i = 0, 1, 2, \dots$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{array}{lcl} x_1' & = & x_1 + x_2 \\ x_2' & = & x_2 - \nu \sin(x_1 + x_2) \end{array} \qquad (\mathrm{mod}\ 2\pi)$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \qquad (\mathrm{mod}\ 2\pi)$$

Tangent map:

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} x'_1 &= x_1 + x_2 \\ x'_2 &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \qquad (\mathrm{mod}\ 2\pi)$$

Tangent map:

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)$$

$$\begin{pmatrix} dx'_1 \\ dx'_2 \end{pmatrix}$$

Example – 2D map

Equations of the map:

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Tangent map:

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)$$

$$\begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\nu\cos(x_1 + x_2) & 1 - \nu\cos(x_1 + x_2) \end{pmatrix}$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{array}{lcl} x_1' & = & x_1 + x_2 \\ x_2' & = & x_2 - \nu \sin(x_1 + x_2) \end{array} \qquad (\mathrm{mod}\ 2\pi)$$

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Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

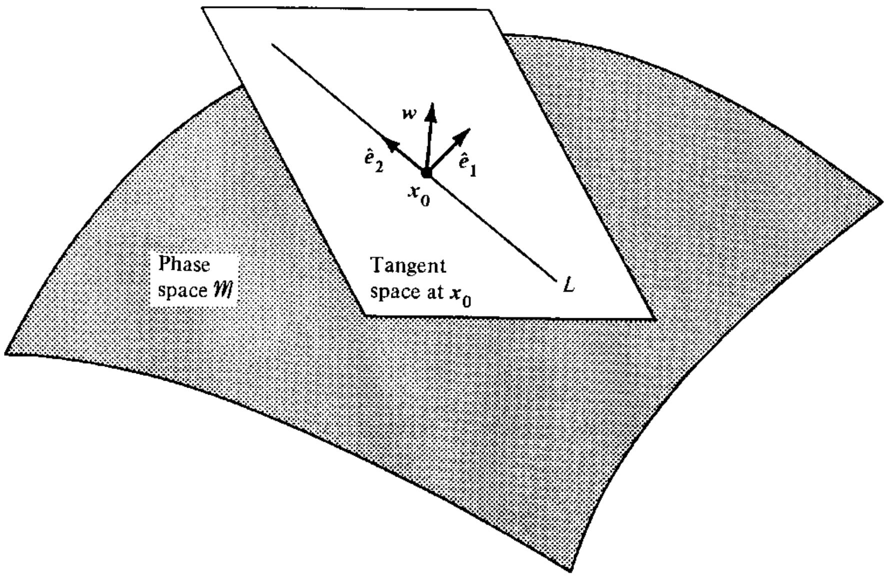
We commonly use the Euclidian norm and set d(0)=||v(0)||=1

Lyapunov Exponents

There exists an **M-dimensional basis** $\{\hat{e}_i\}$ of v such that for any v, σ takes one of the M (possibly nondistinct) values

$$\sigma_i(x(0)) = \sigma(x(0), \hat{e}_i)$$

which are the **Lyapunov exponents.**



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

In autonomous Hamiltonian systems the M exponents are ordered in pairs of opposite sign numbers and two of them are 0.

Computation of the Maximum Lyapunov Exponent

Due to the exponential growth of v(t) (and of d(t)=||v(t)||) we renormalize v(t) from time to time.

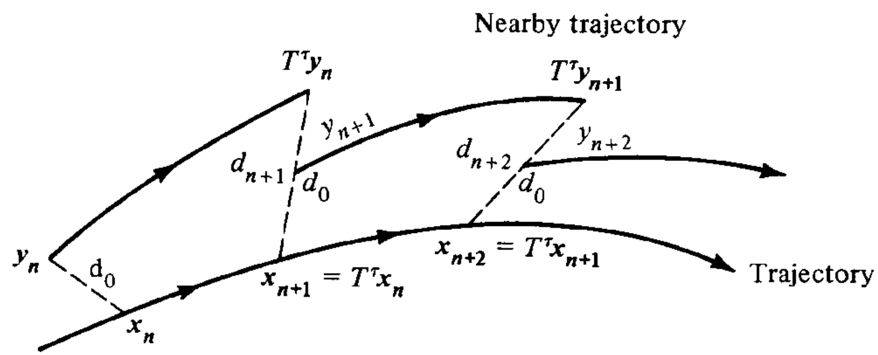


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximum Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i$$

Maximum Lyapunov Exponent

σ_1=0: Regular motion

$\sigma_1 \neq$0: Chaotic motion

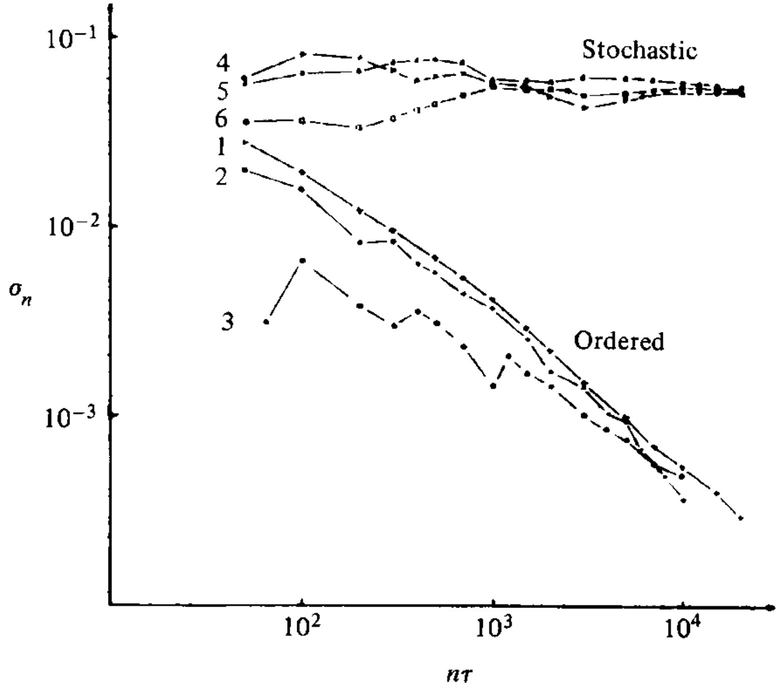
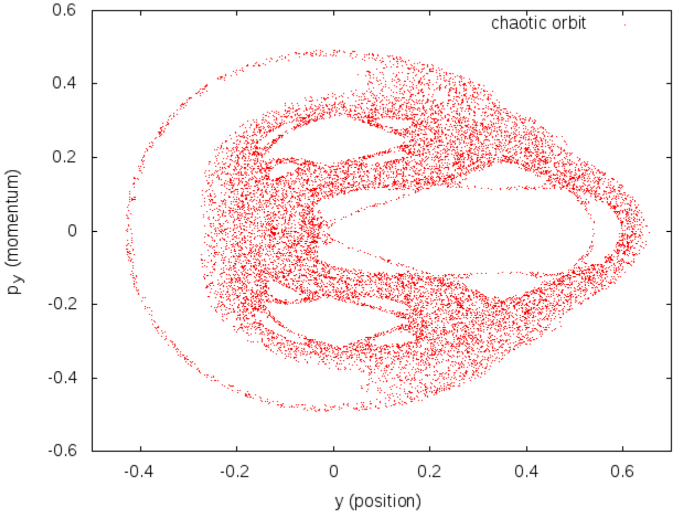


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al*., 1976).

If we start with more than one linearly independent deviation vectors they will align to the direction defined by the largest Lyapunov exponent for chaotic orbits.

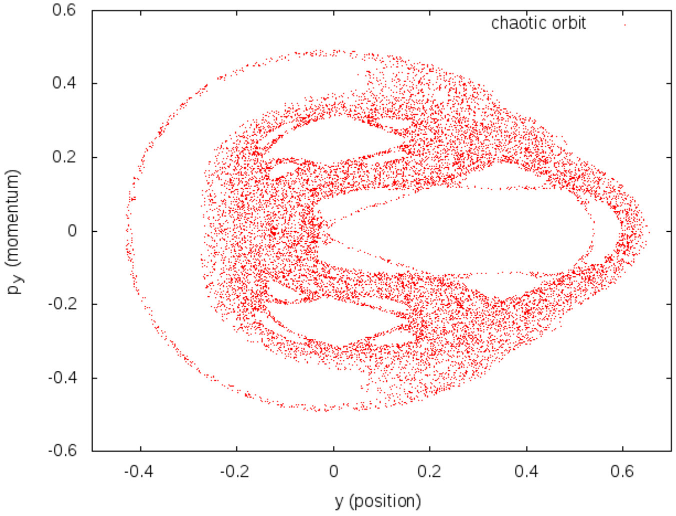
Maximum Lyapunov Exponent

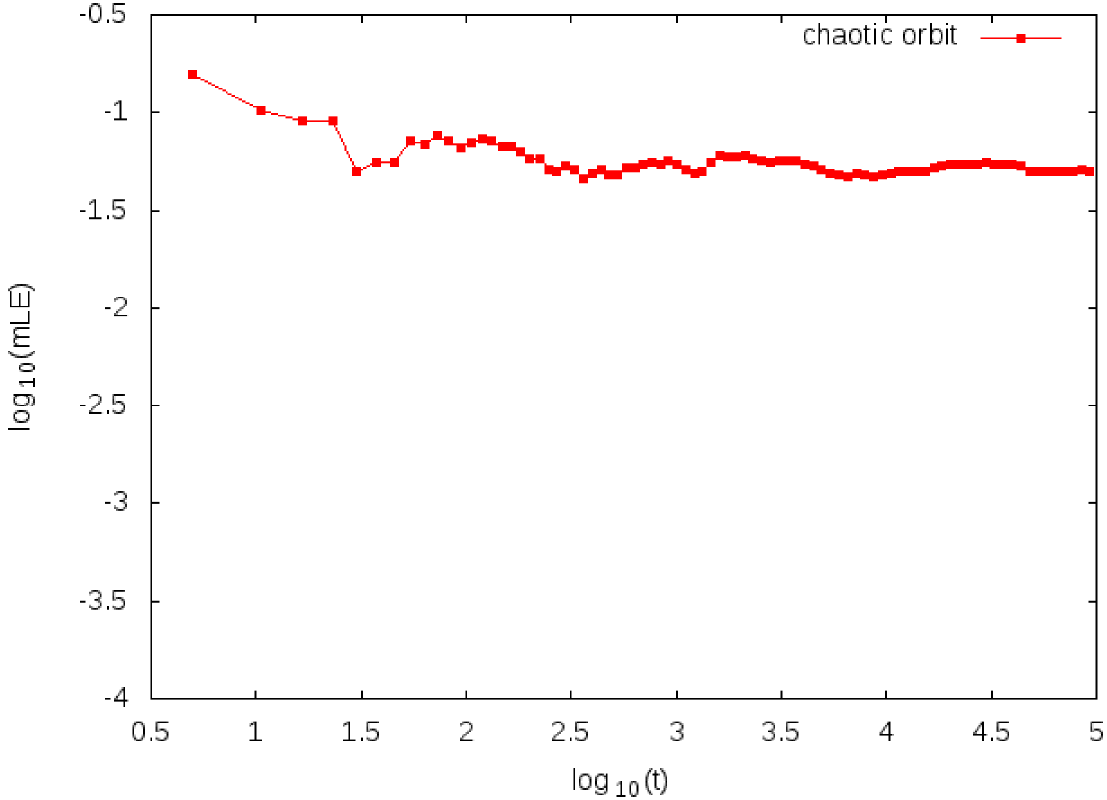
Hénon-Heiles system: Chaotic orbit



Maximum Lyapunov Exponent

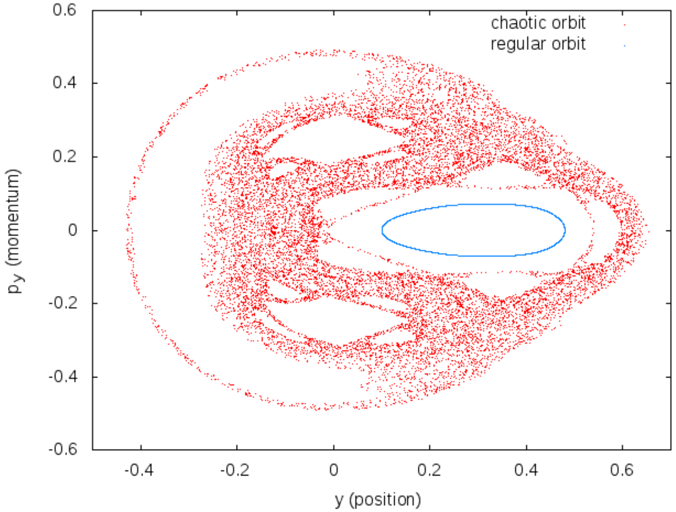
Hénon-Heiles system: Chaotic orbit

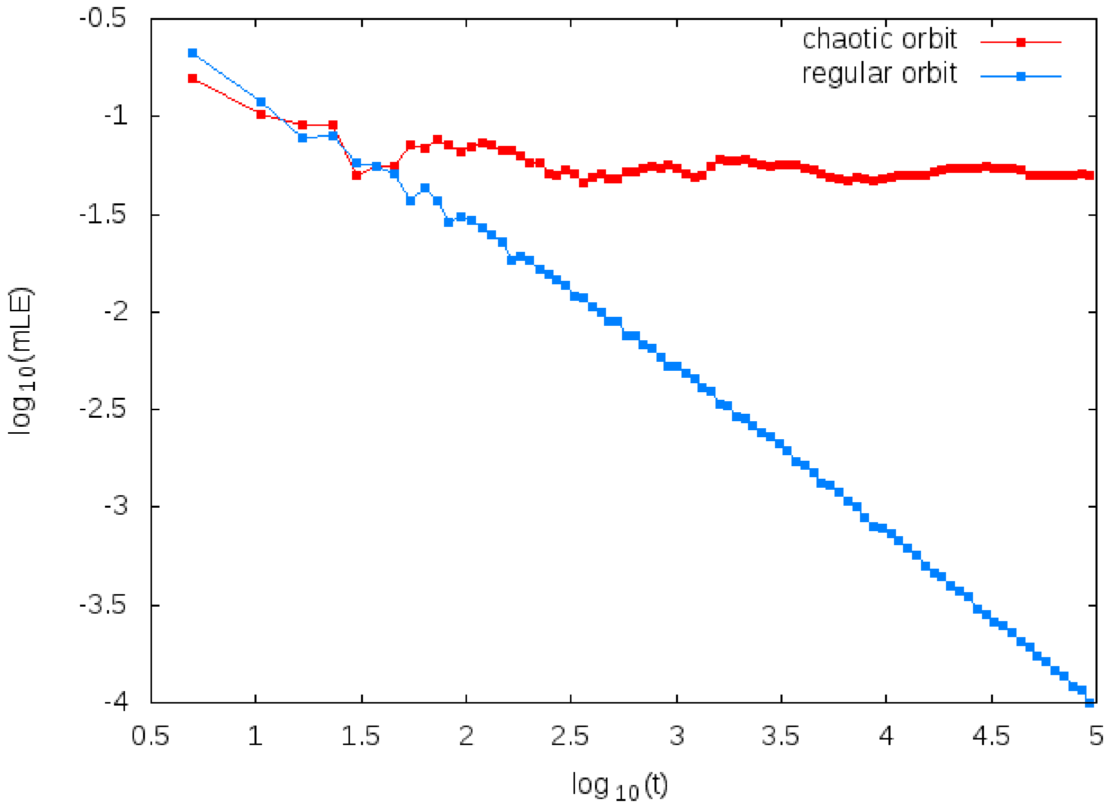




Maximum Lyapunov Exponent

Hénon-Heiles system: Chaotic orbit and Regular orbit





The
Smaller ALignment Index
(SALI)
method

Definition of the SALI

We follow the evolution in time of <u>two different initial deviation vectors</u> ($v_1(0)$, $v_2(0)$), and define the SALI (Ch.S. 2001, J. Phys. A) as:

$$SALI(t) = \min\left\{ \left\| \hat{v}_1(t) + \hat{v}_2(t) \right\|, \left\| \hat{v}_1(t) - \hat{v}_2(t) \right\| \right\}$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

When the two vectors become collinear

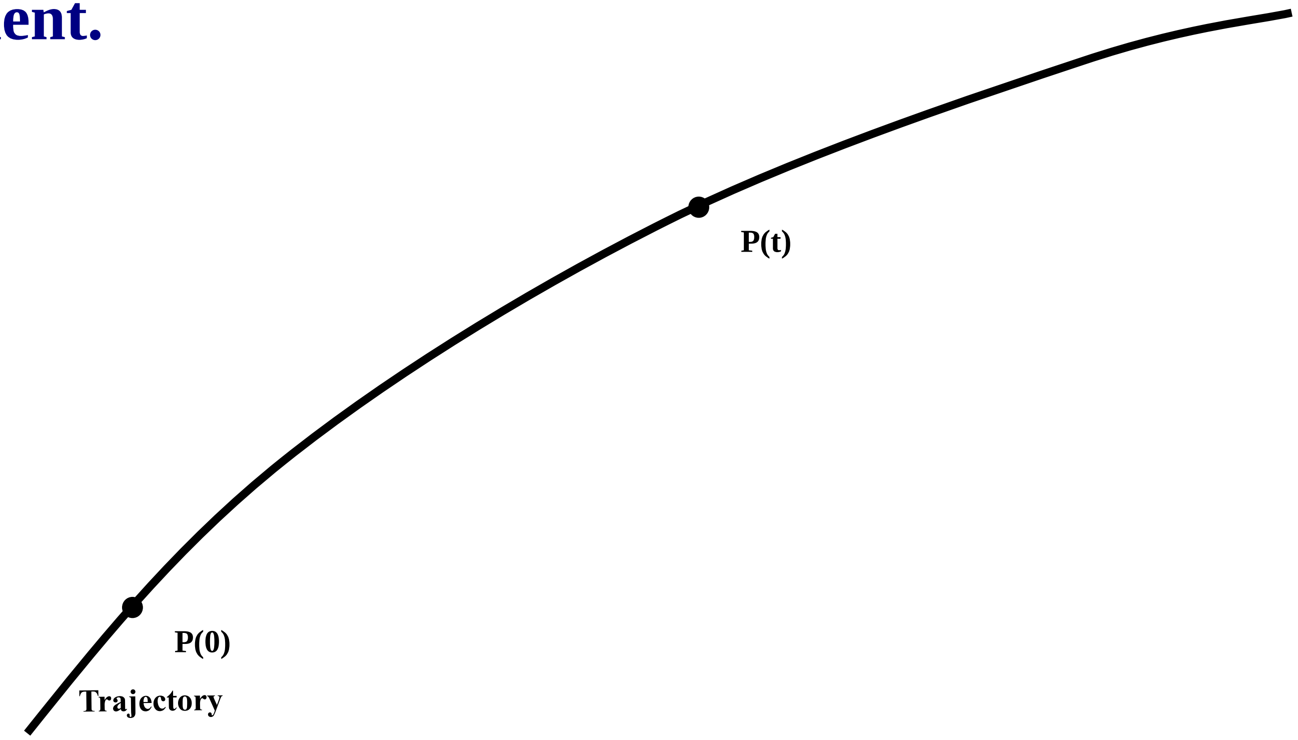
$$SALI(t) \rightarrow 0$$

Behavior of the SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximum Lyapunov exponent.

Behavior of the SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximum Lyapunov exponent.



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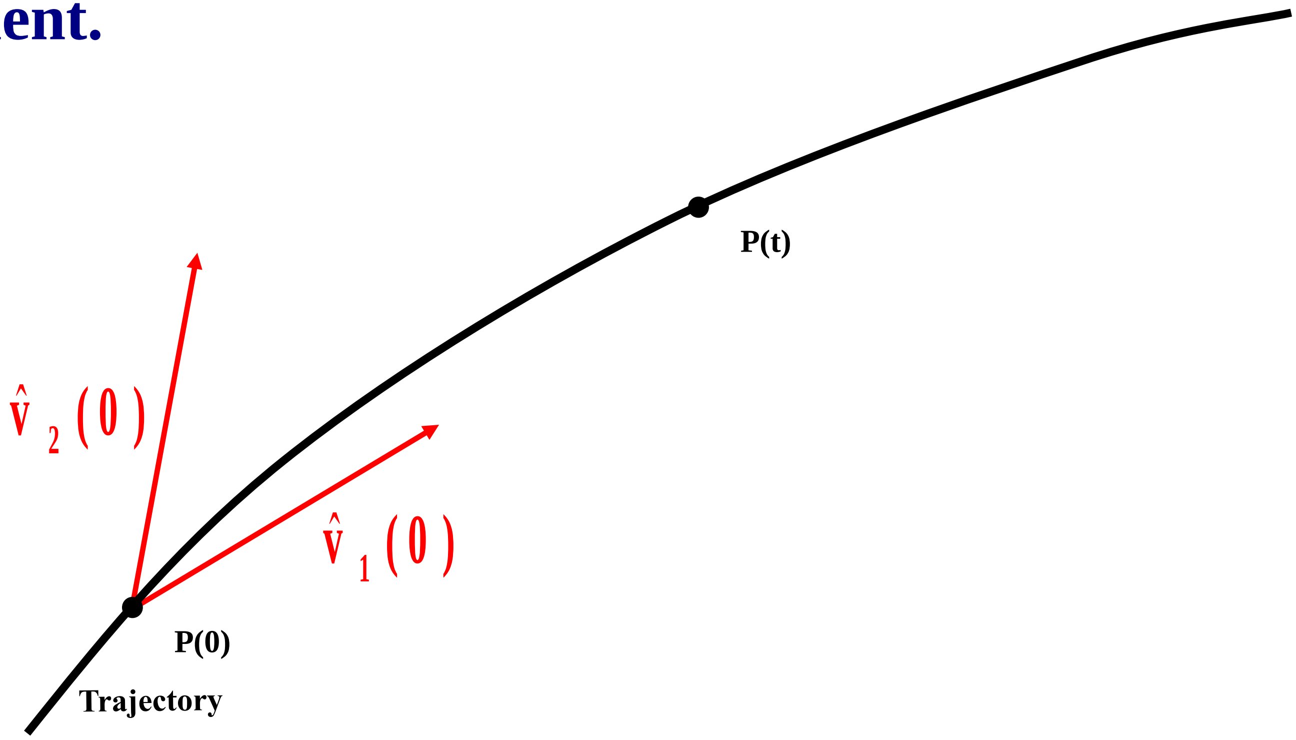

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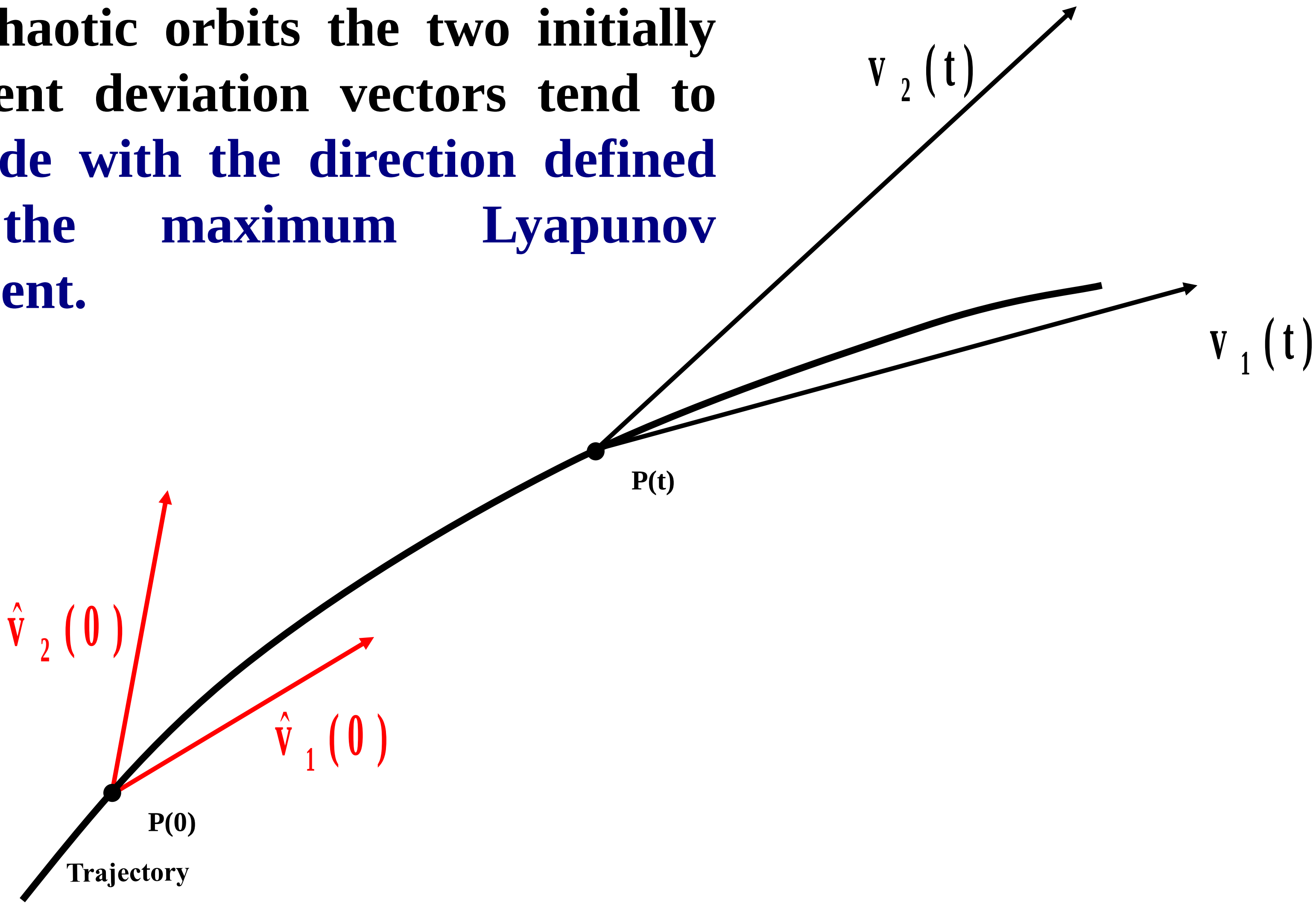

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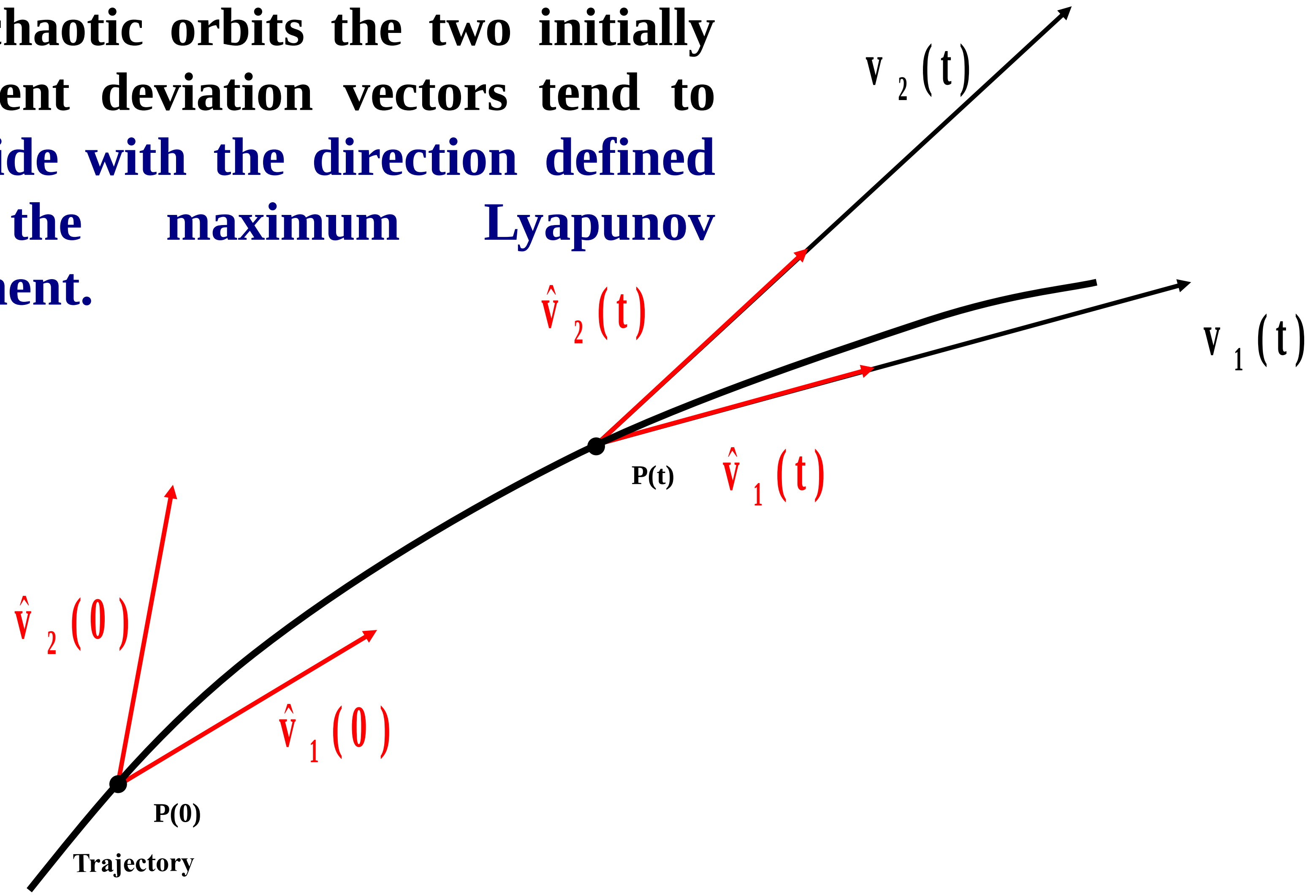

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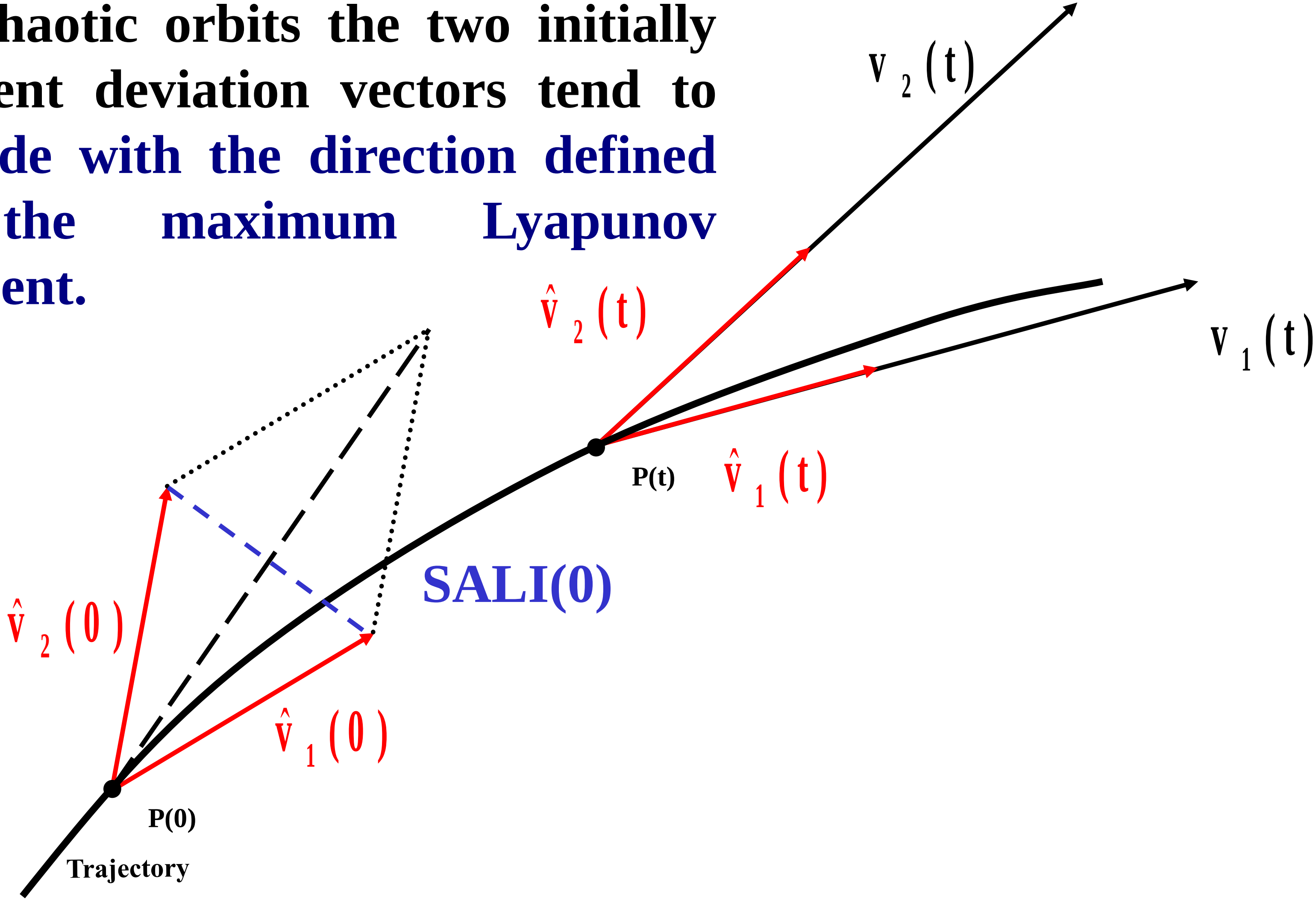

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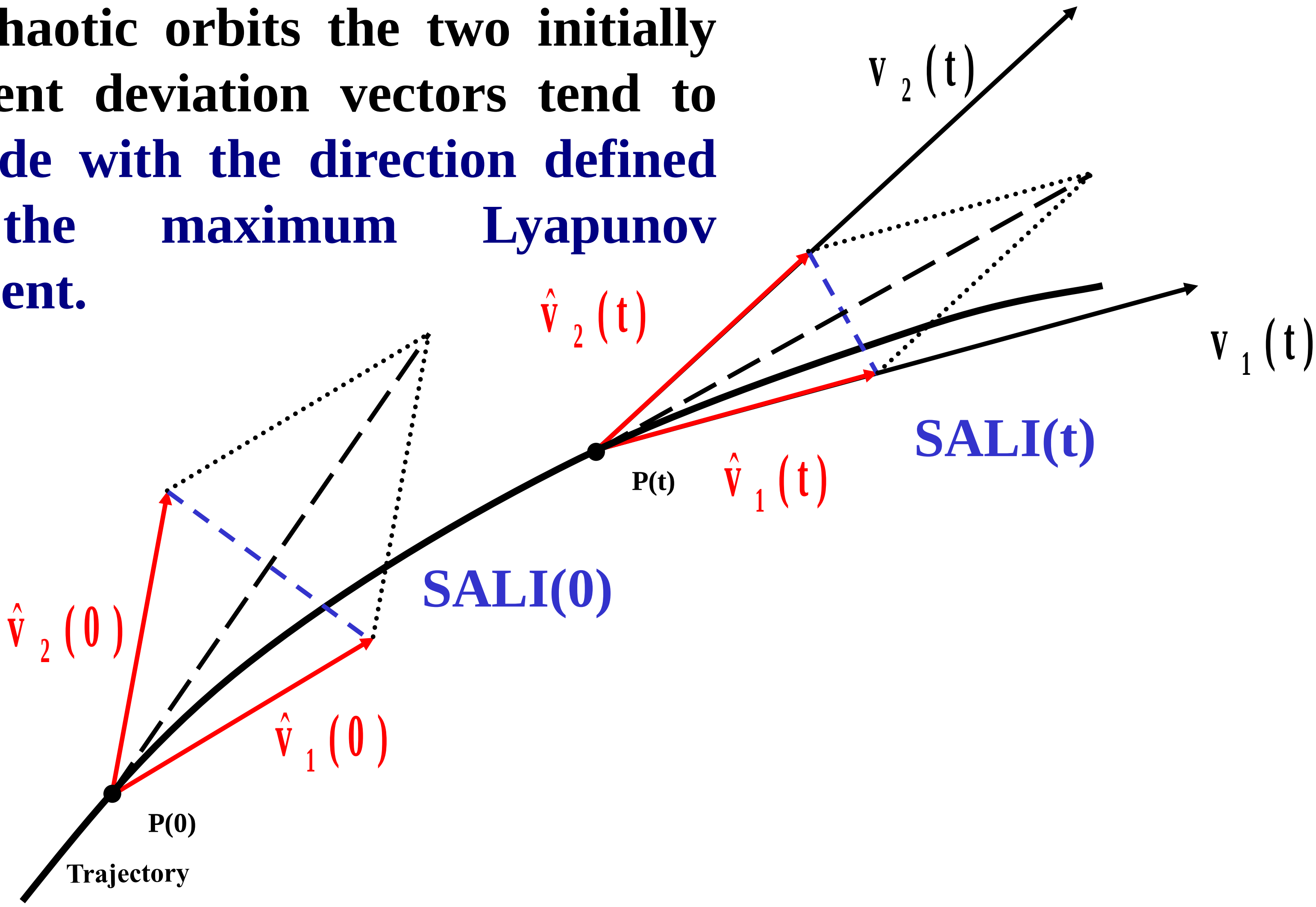
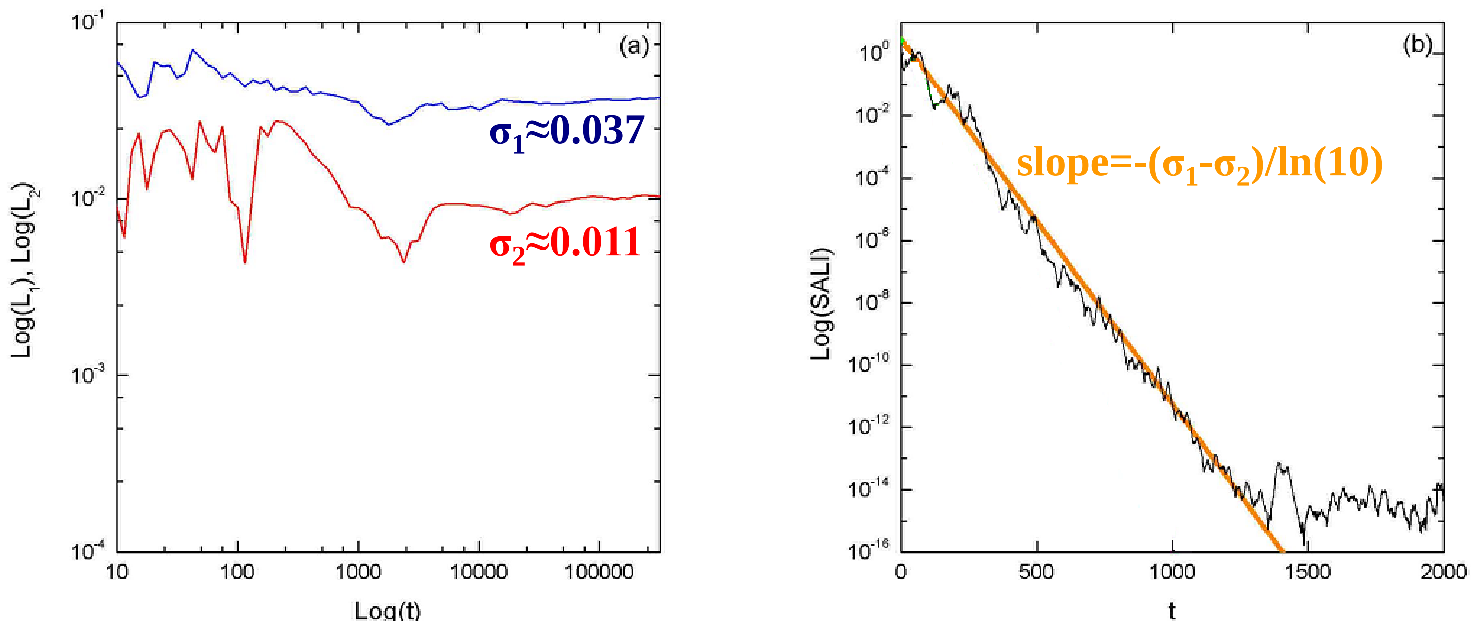


Behavior of the SALI for chaotic motion

We test the validity of the approximation $SALI \sim e^{-(\sigma 1-\sigma 2)t}$ (Ch.S., Antonopoulos, Bountis, Vrahatis, 2004, J. Phys. A) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=1.4142$, $\omega_3=1.7321$, $H=0.09$



Behavior of the SALI for regular motion

Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.

Behavior of the SALI for regular motion

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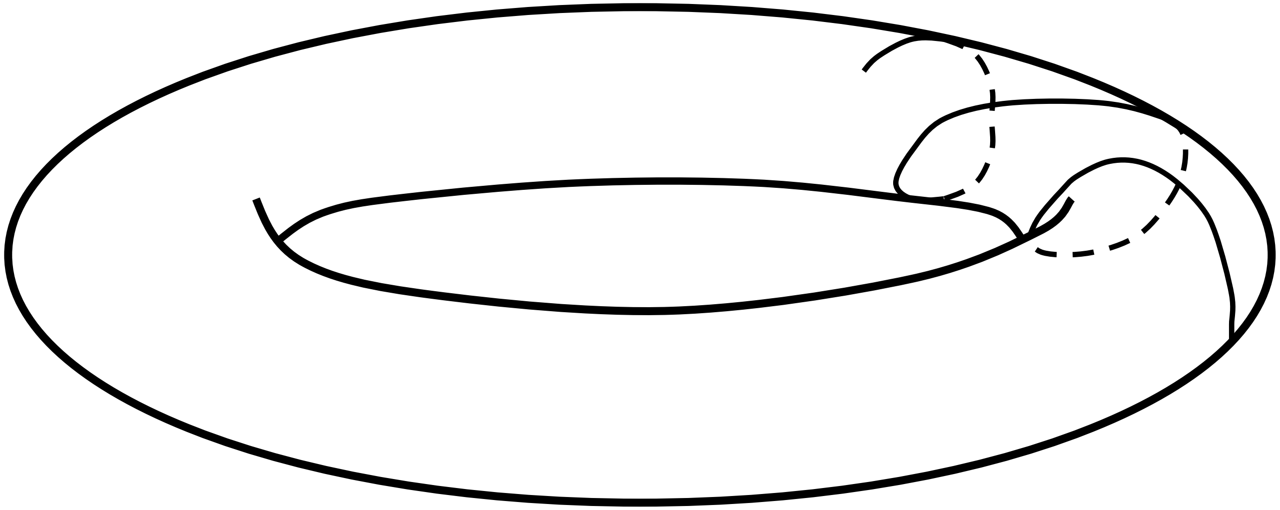

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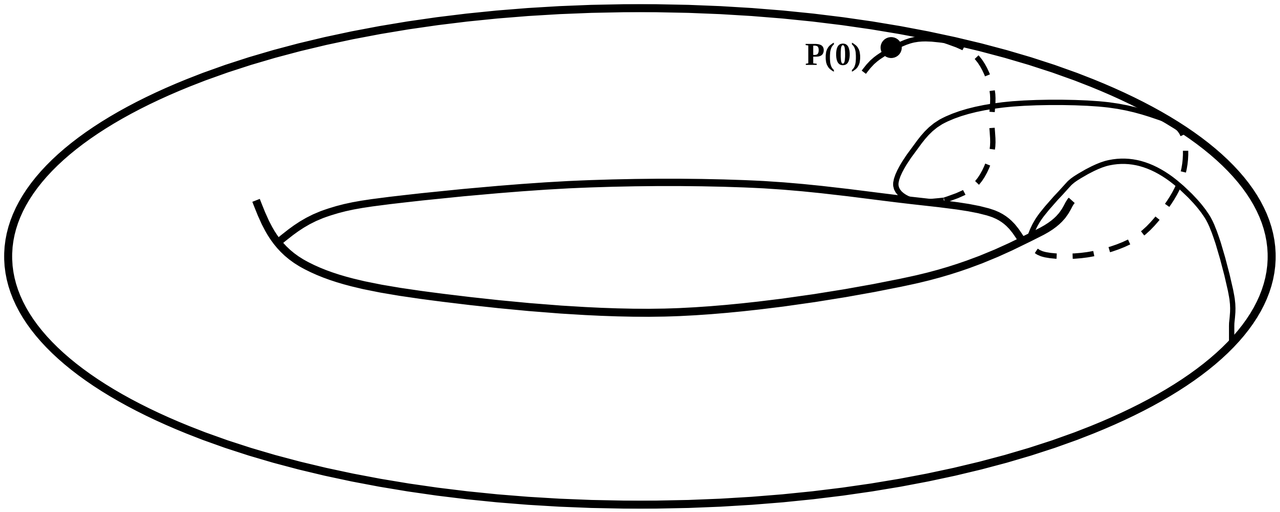

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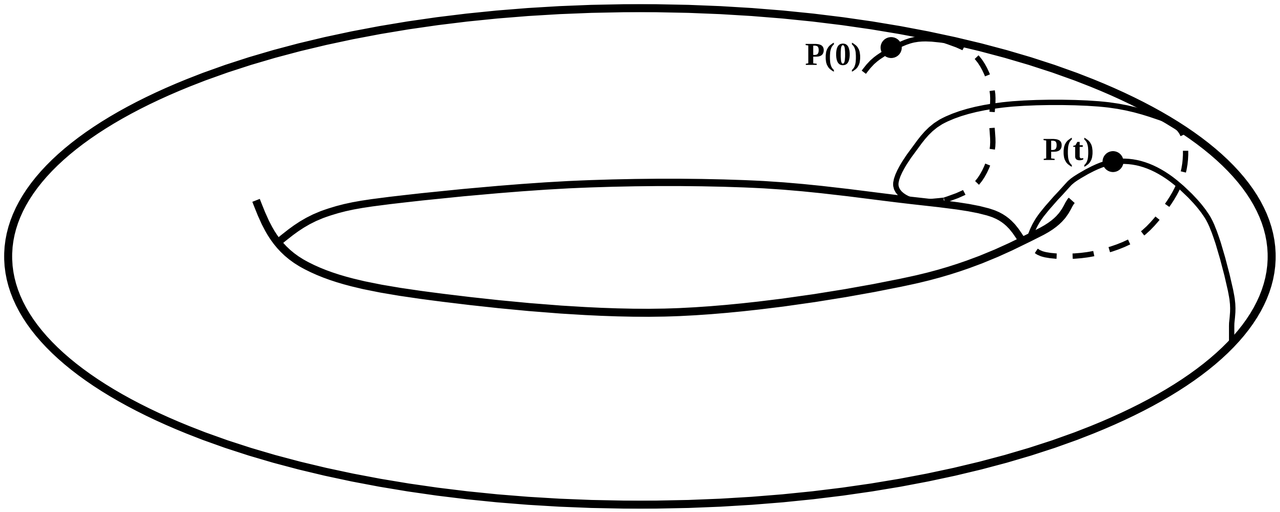

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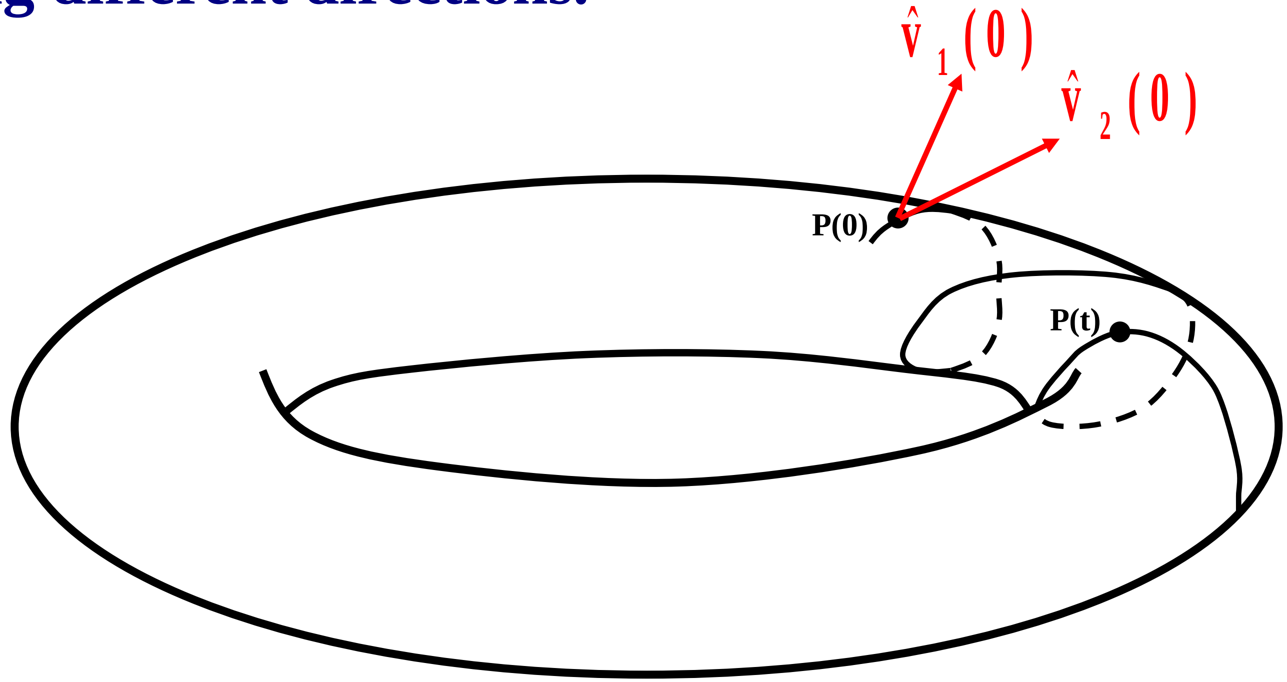

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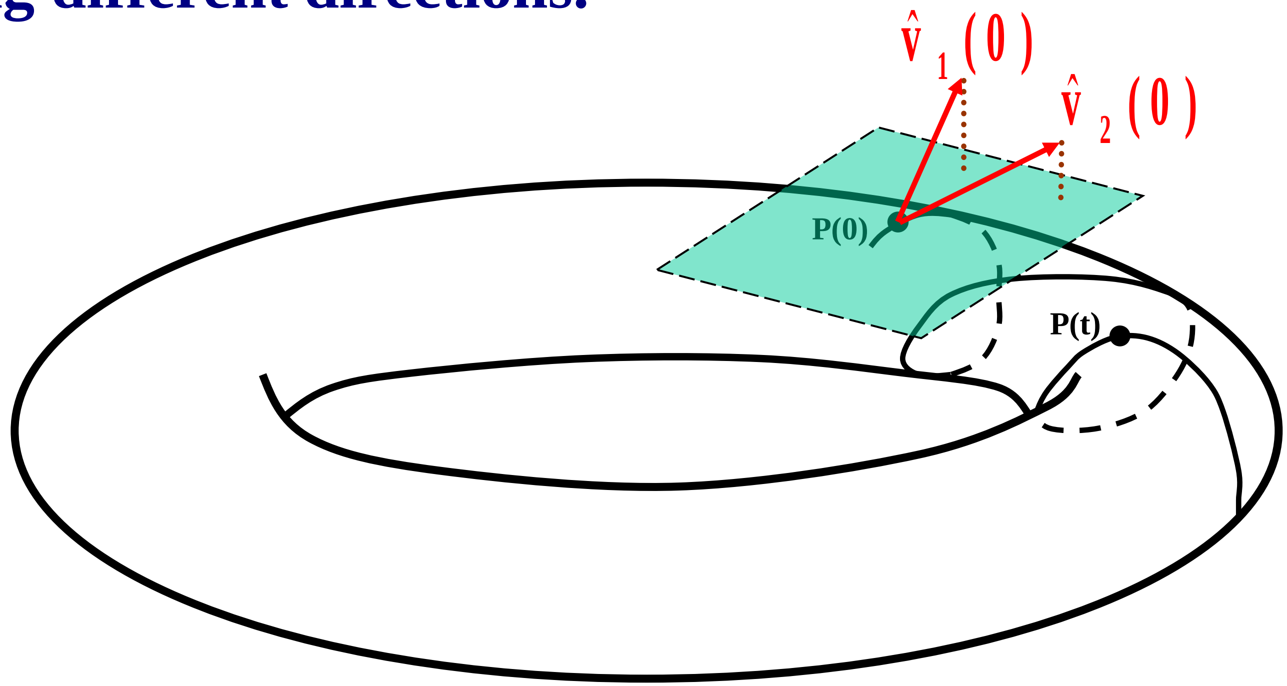

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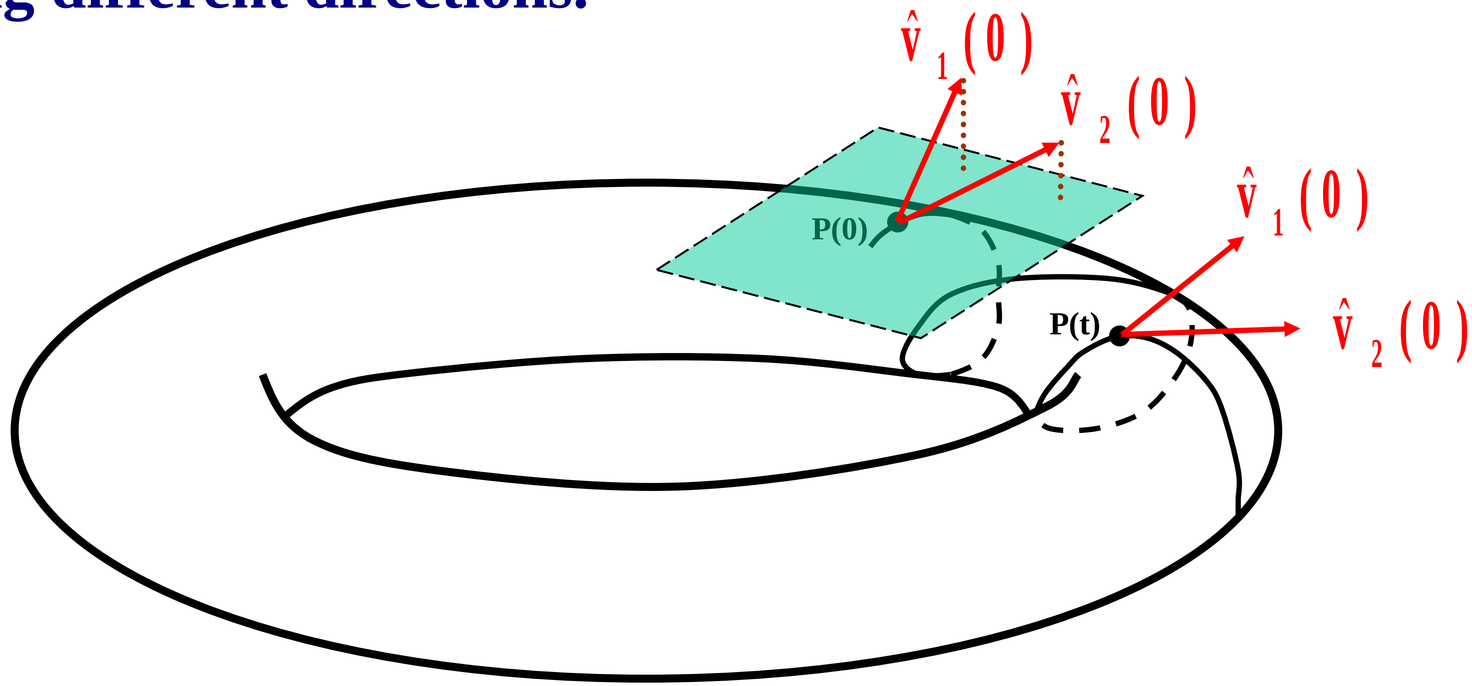

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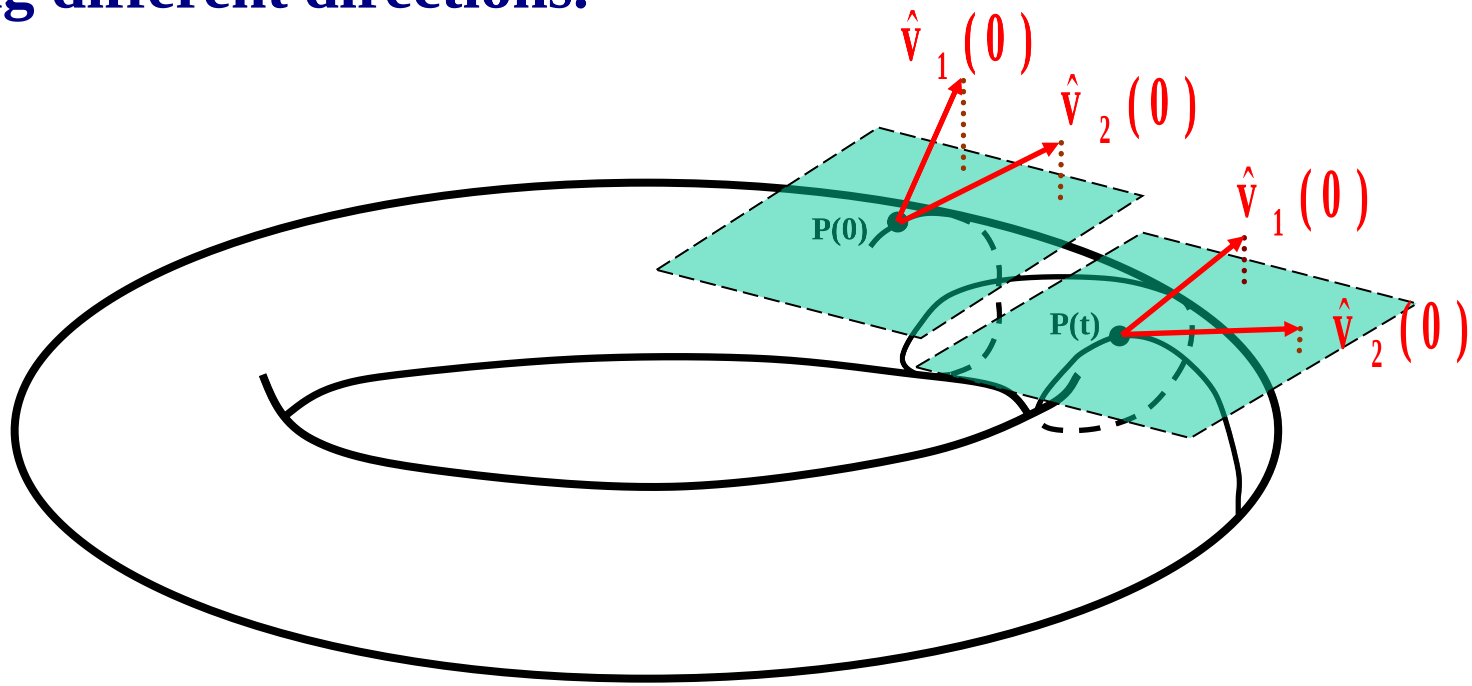
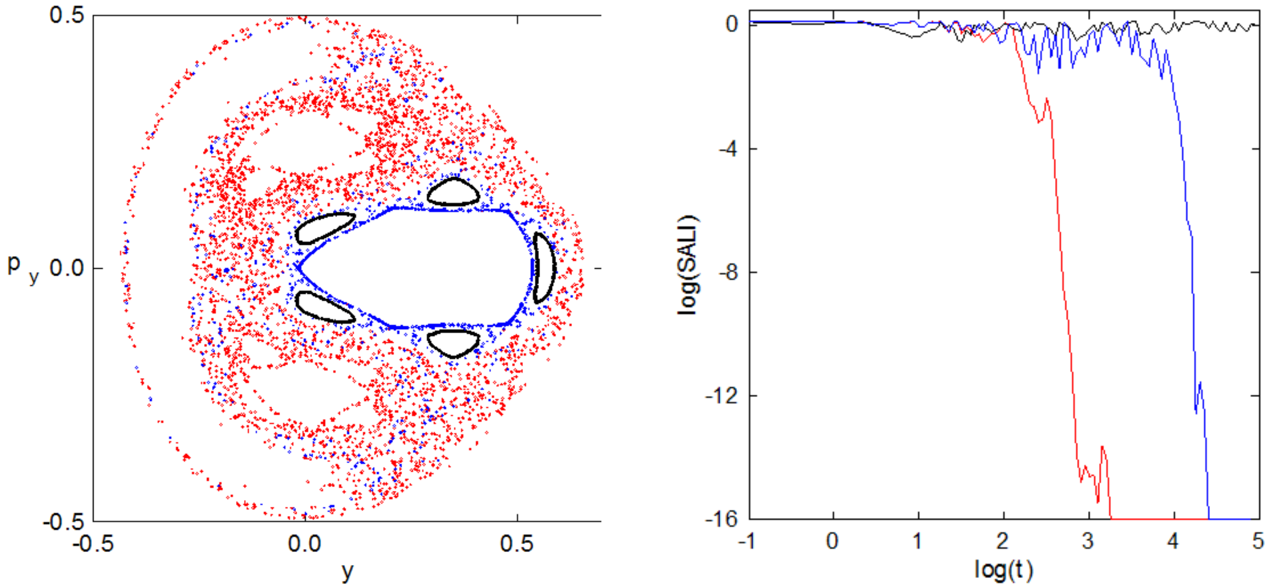


Applications – Hénon-Heiles system

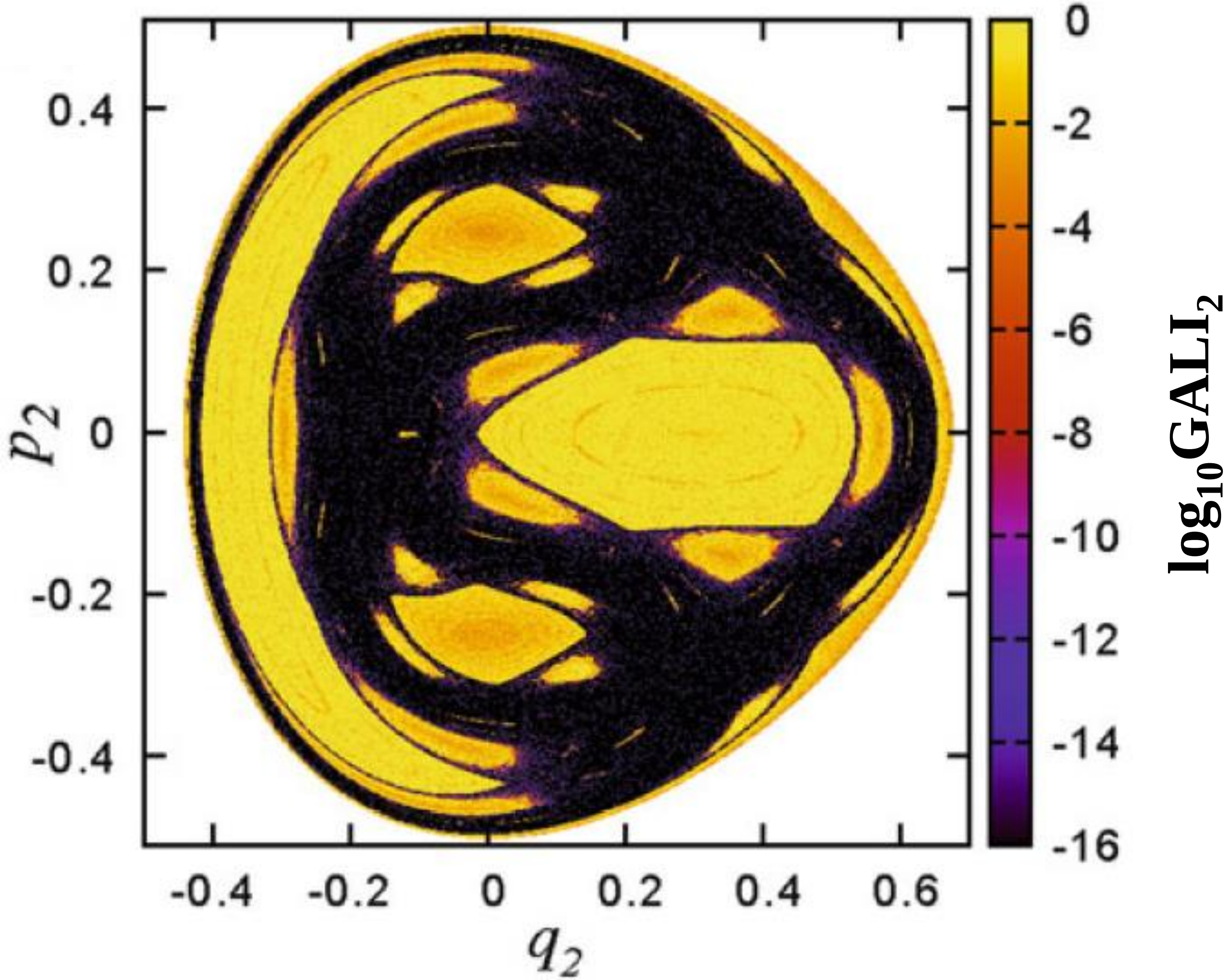
As an example, we consider the 2D Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

For E=1/8 we consider the orbits with initial conditions:
Regular orbit, x=0, y=0.55, p_x=0.2417, p_y=0
Chaotic orbit, x=0, y=-0.016, p_x=0.49974, p_y=0
Chaotic orbit, x=0, y=-0.01344, p_x=0.49982, p_y=0

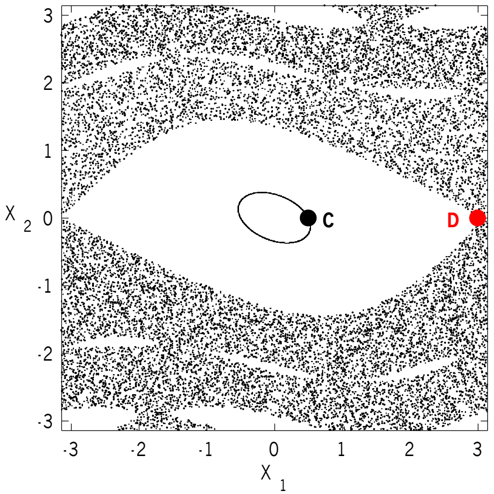


Applications – Hénon-Heiles system



Applications – 4D map

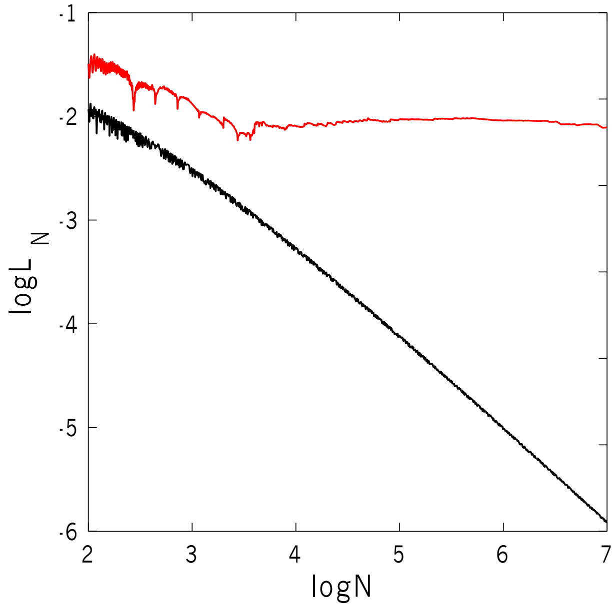
$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \end{aligned} \quad (\text{mod } 2\pi)$$



For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

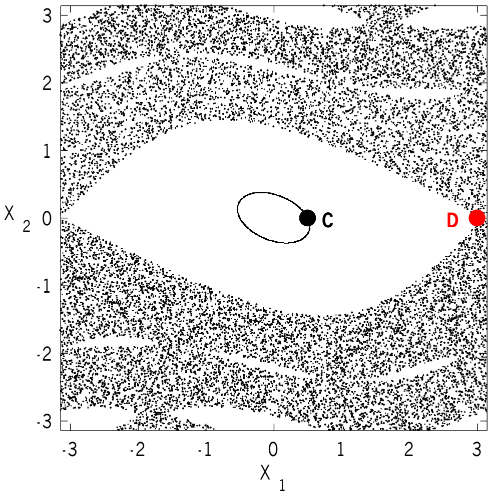
regular orbit C *with initial conditions* $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D *with initial conditions* $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



Applications – 4D map

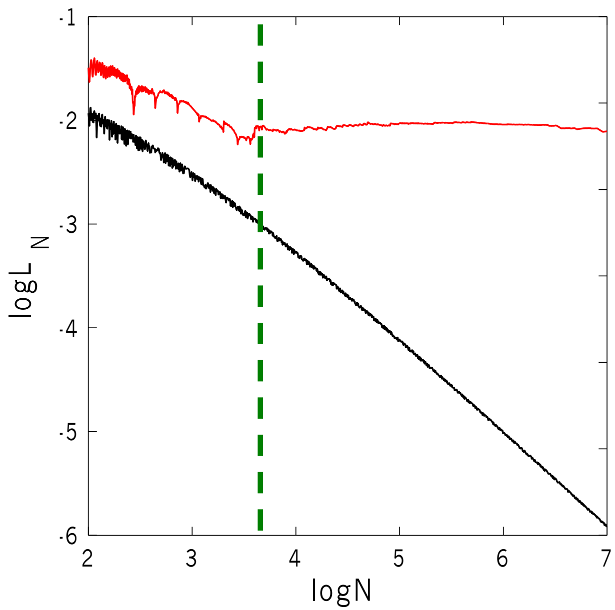
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\quad (\text{mod } 2\pi)
$$



For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

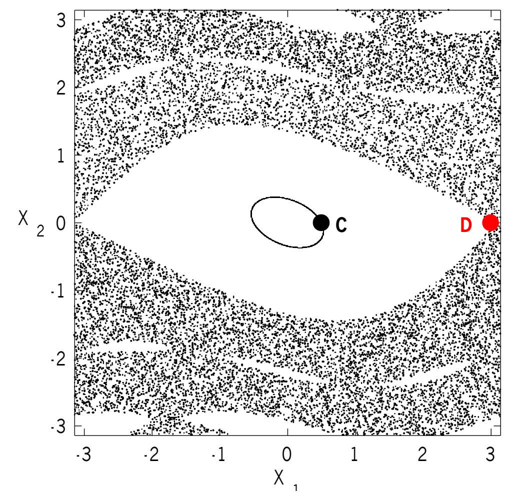
regular orbit C *with initial conditions* $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D *with initial conditions* $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



Applications – 4D map

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \end{aligned} \quad (\text{mod } 2\pi)$$



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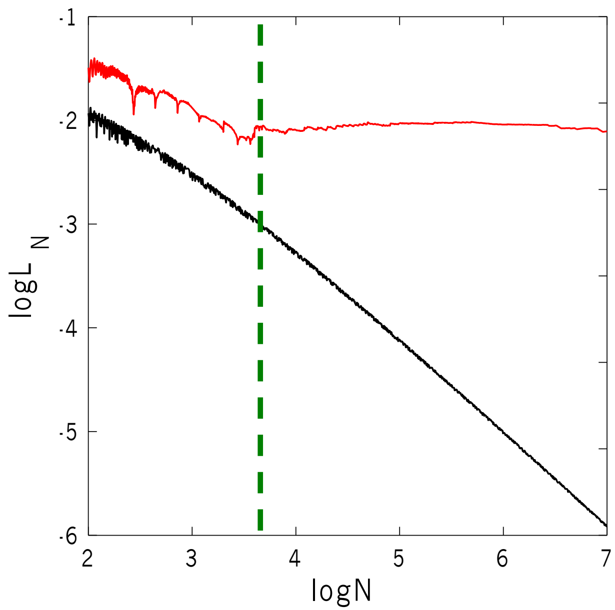

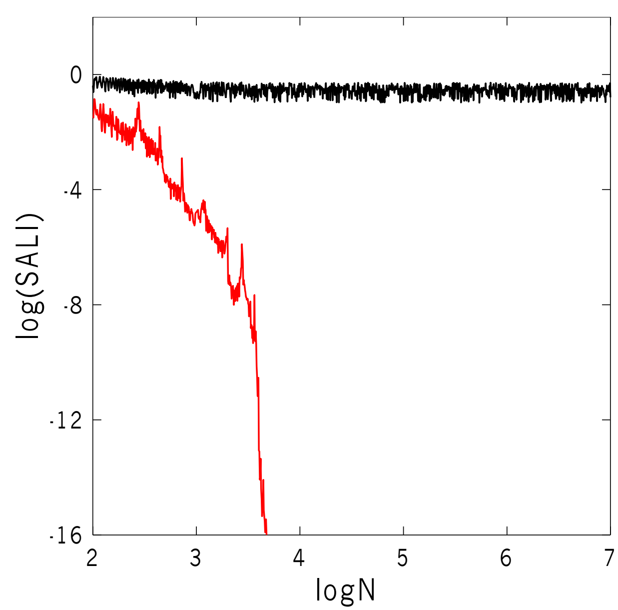

Applications – 4D map

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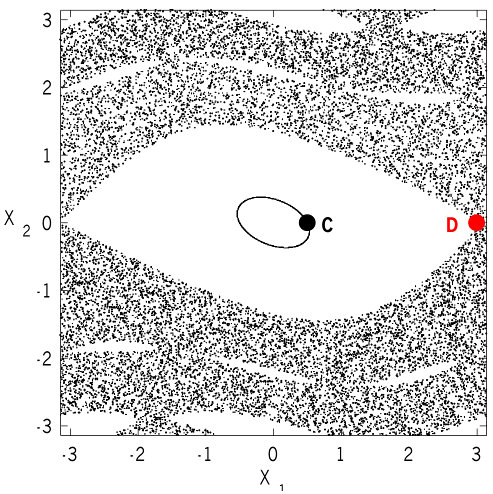

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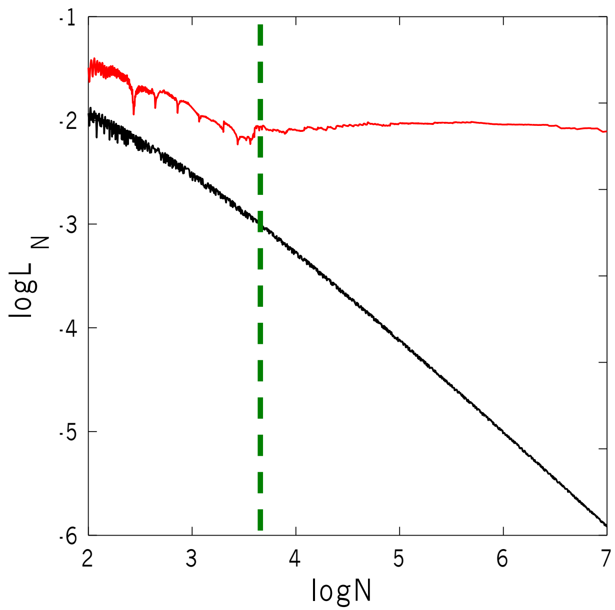

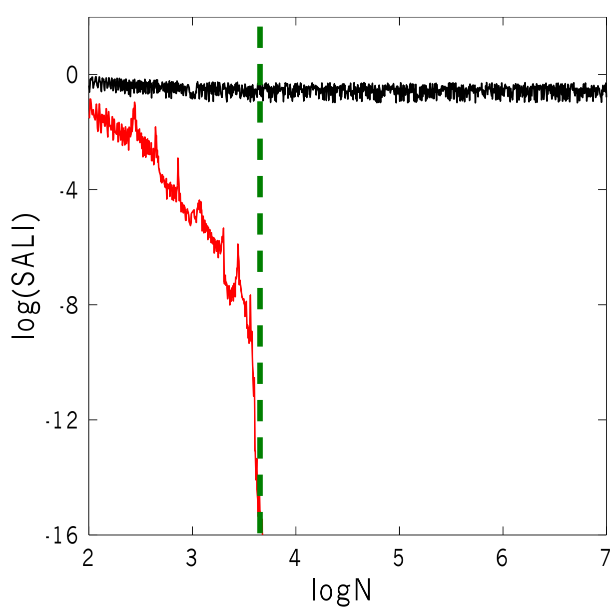

The
Generalized ALignment Indices
(GALIs)
method

Definition of the Generalized Alignment Index (GALI)

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.

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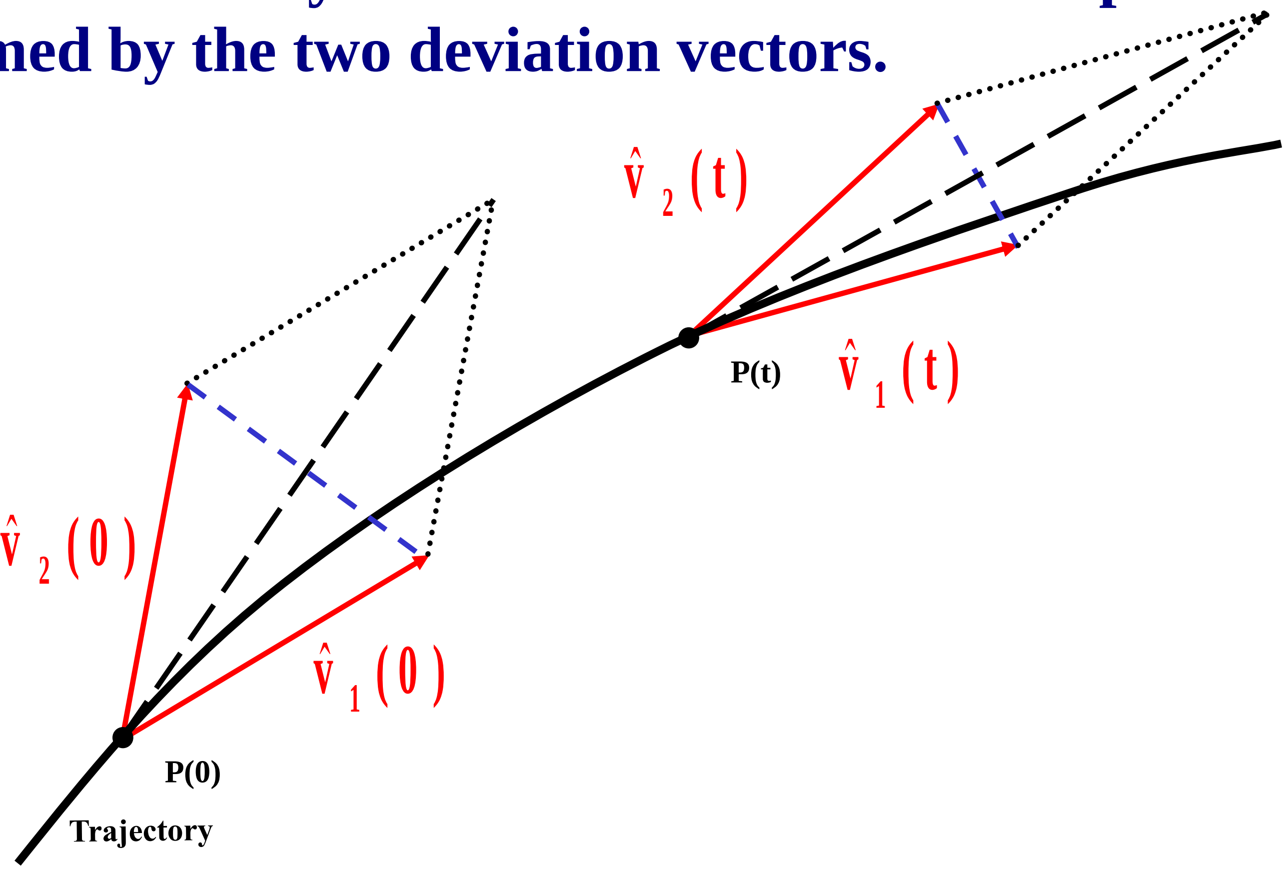

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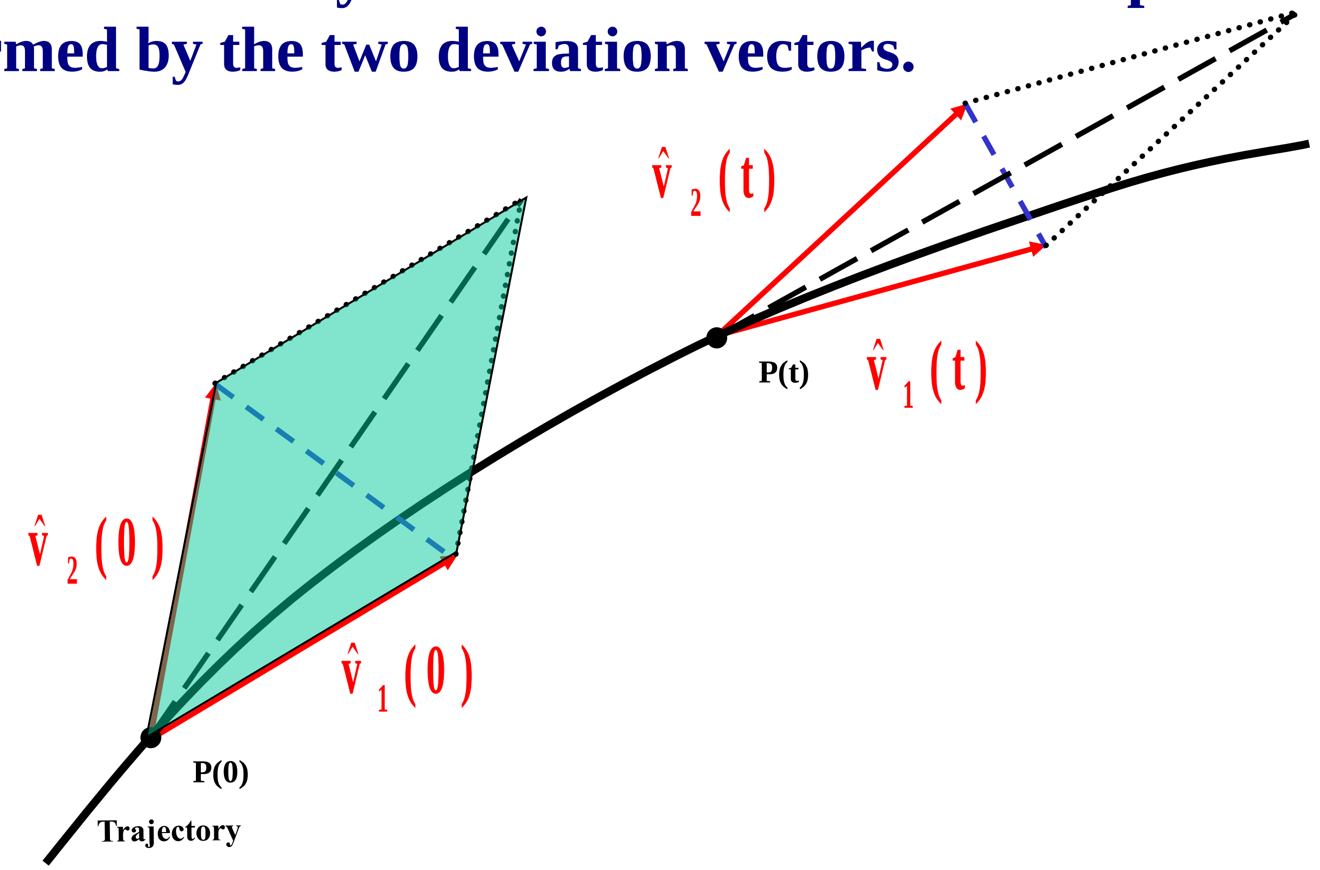

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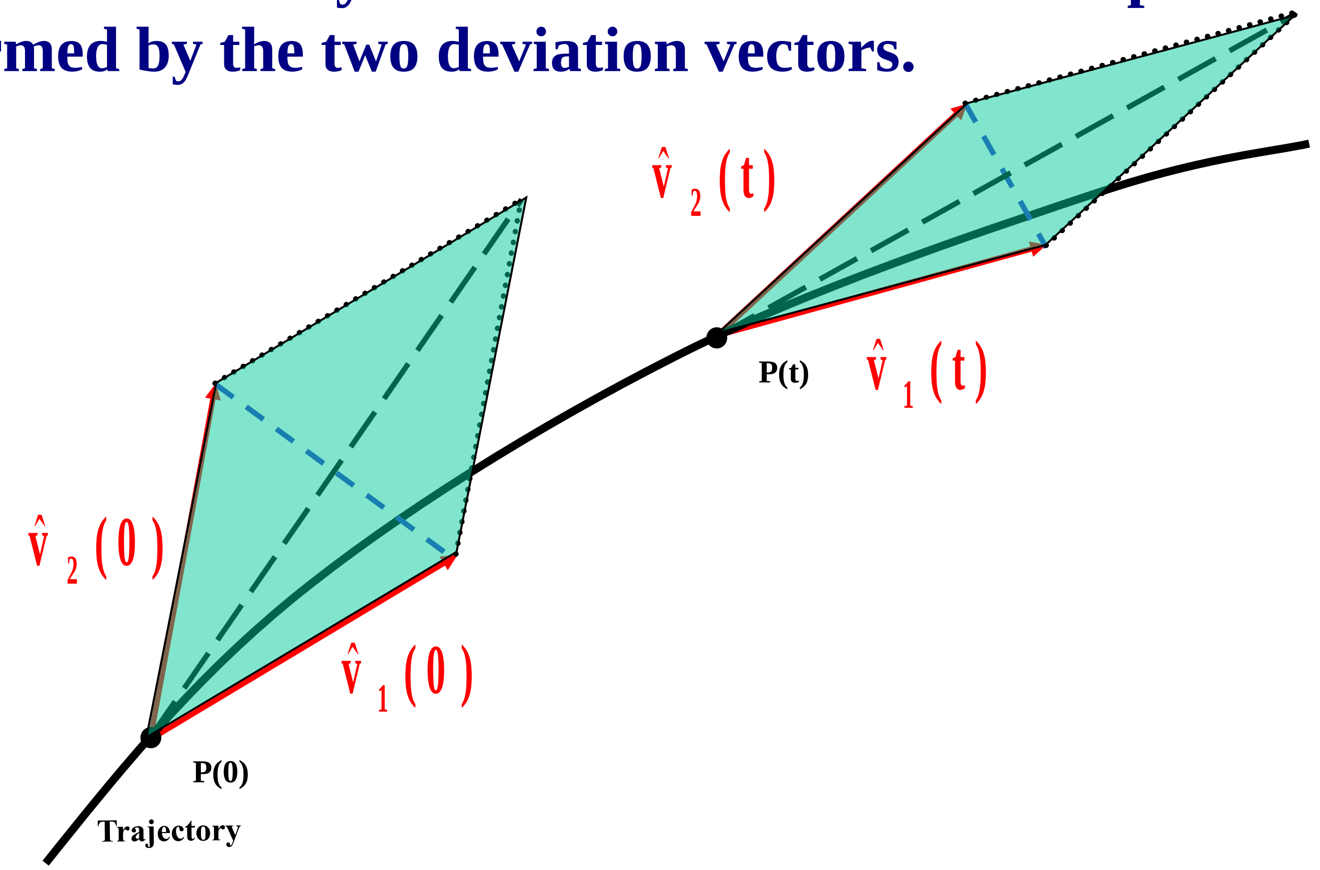

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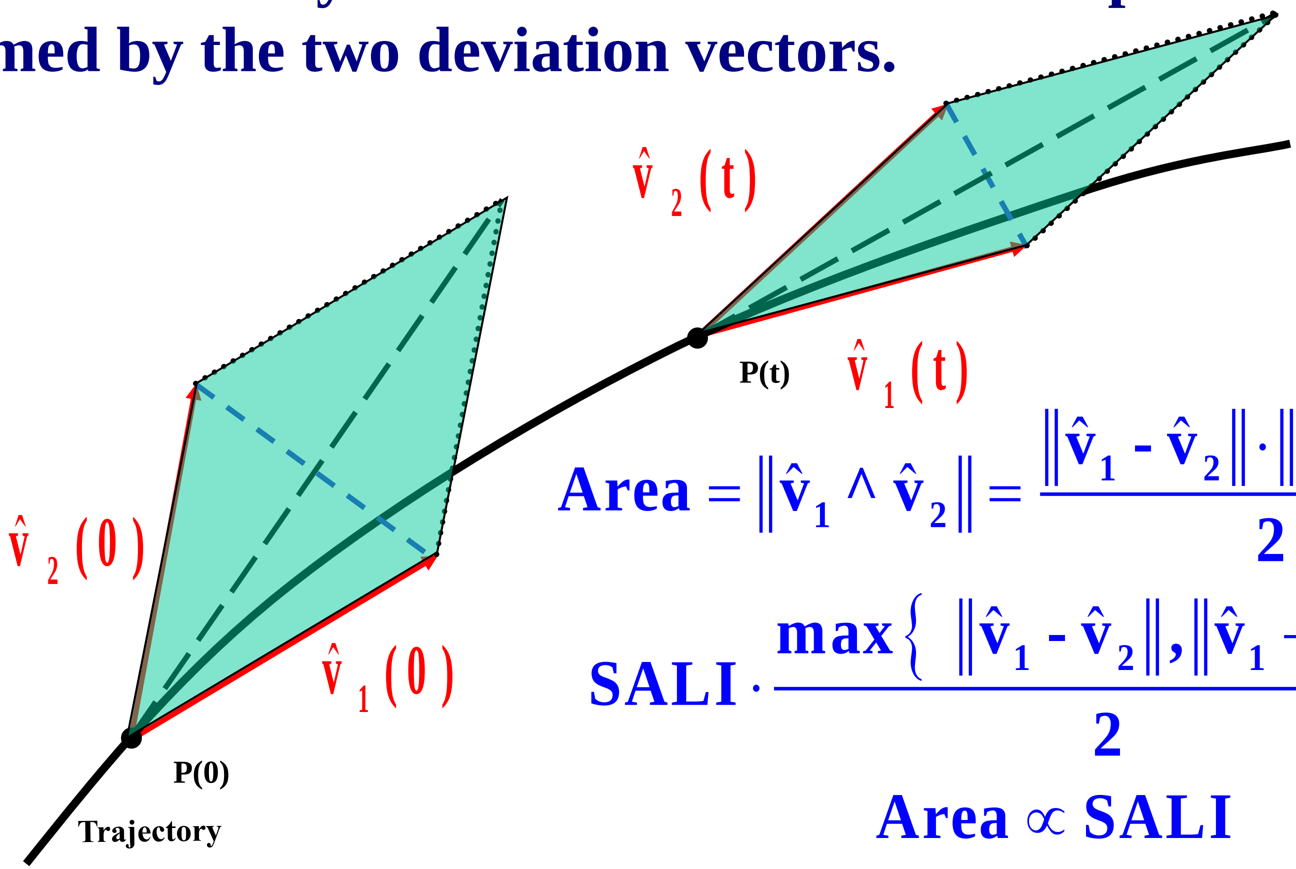


$$\mathbf{Area} = \left\|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2\right\| = \frac{\left\|\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2\right\| \cdot \left\|\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2\right\|}{2} =$$

$$\mathbf{SALI} \cdot \frac{\mathbf{max}\left\{ \left\|\hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2\right\|, \left\|\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2\right\| \right\}}{2} \Rightarrow$$

$$\mathbf{Area} \propto \mathbf{SALI}$$

Definition of the GALI

In the case of an N degree of freedom Hamiltonian system or a 2N symplectic map we follow the evolution of

k deviation vectors with $2 \leq k \leq 2N$,

and define (Ch.S., Bountis, Antonopoulos, 2007, Physica D) the Generalized Alignment Index (GALI) of order k :

$$GALI_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t) \right\|$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

Behavior of the $GALI_k$ for chaotic motion

$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$GALI_k(t) \propto e^{-\left[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)\right]t}$$

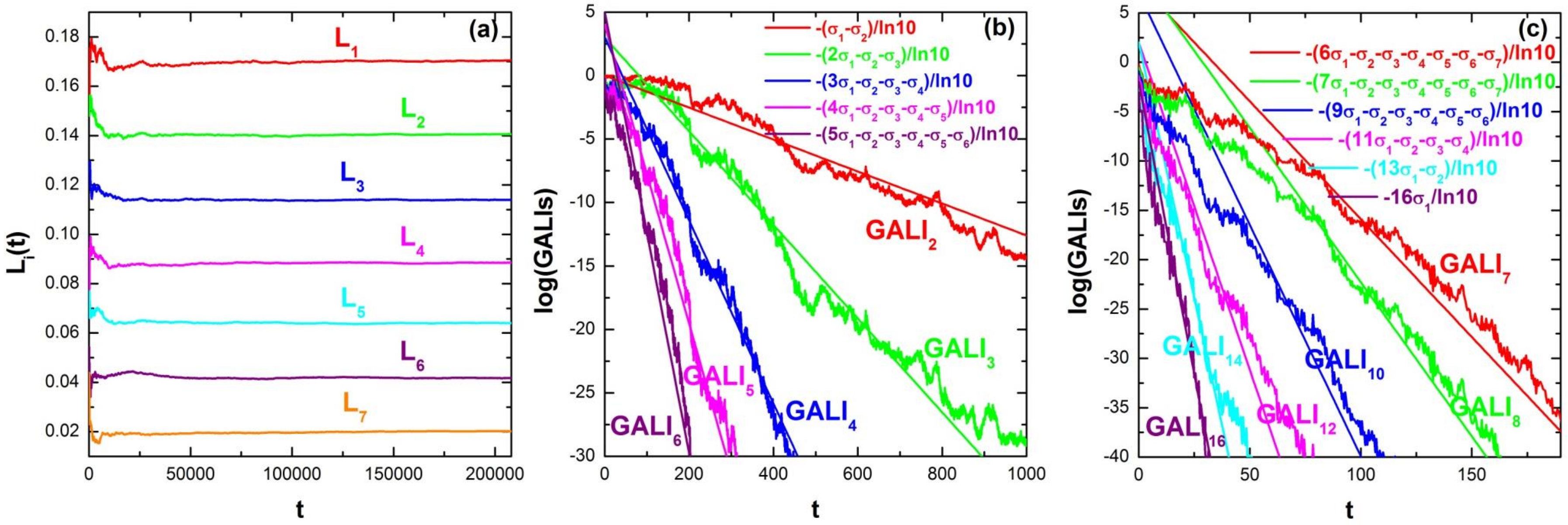
The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of the $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=0}^{N}\left[\frac{1}{2}(q_{i+1} - q_i)^2 + \frac{\beta}{4}(q_{i+1} - q_i)^4\right]$$

with fixed boundary conditions, N=8 and β=1.5.



Behavior of the $GALI_k$ for regular motion

If the motion occurs on an s-dimensional torus with s≤N then the behavior of $GALI_k$ is given by (Ch.S., Bountis, Antonopoulos, 2008, Eur. Phys. J. Sp. Top.):

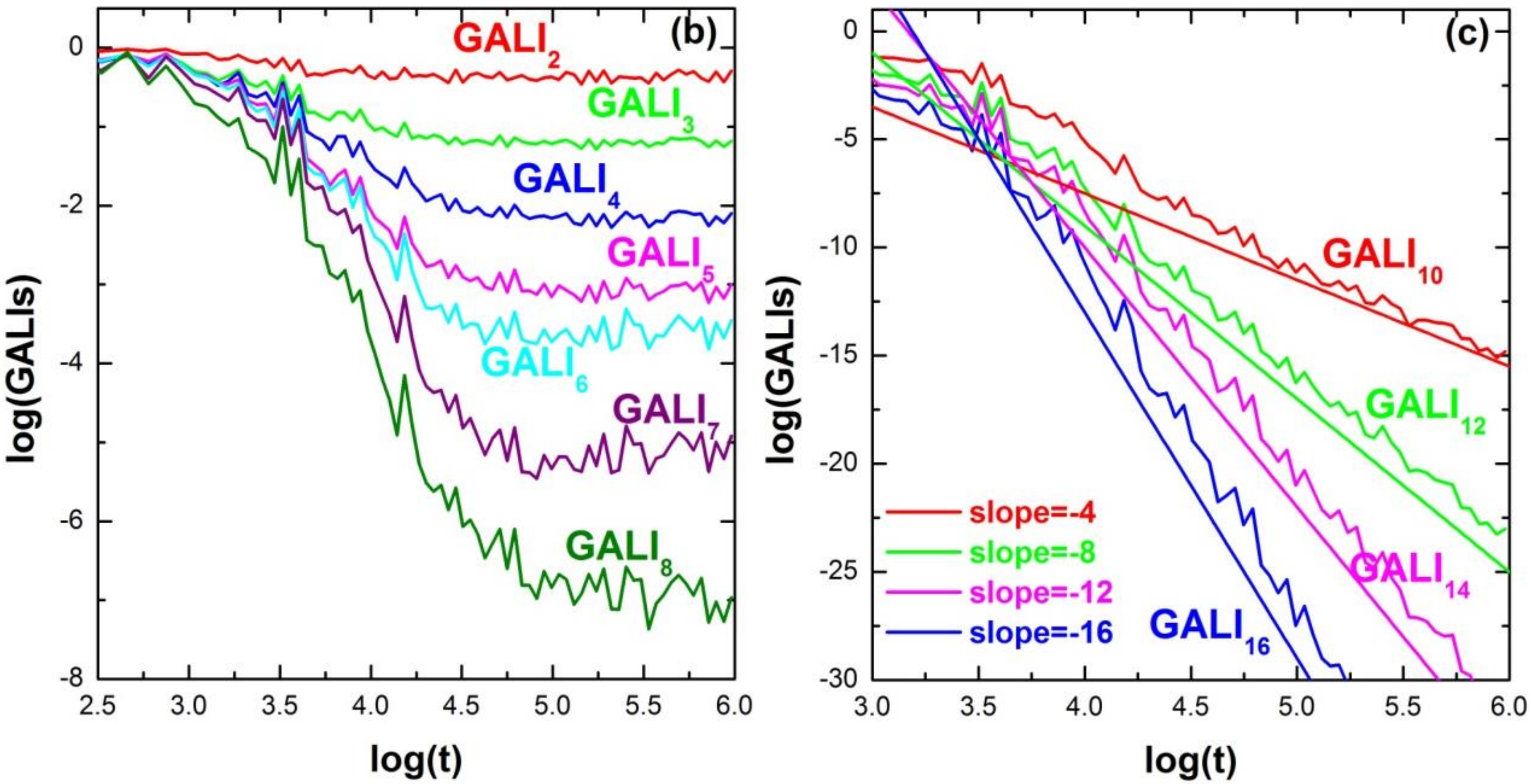
$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \le k \le s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \le 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \le 2N \end{cases}$$

while in the common case with s=N we have :

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \le k \le N \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad N < k \le 2N \end{cases}$$

Behavior of the $GALI_k$ for regular motion

N=8 FPU system

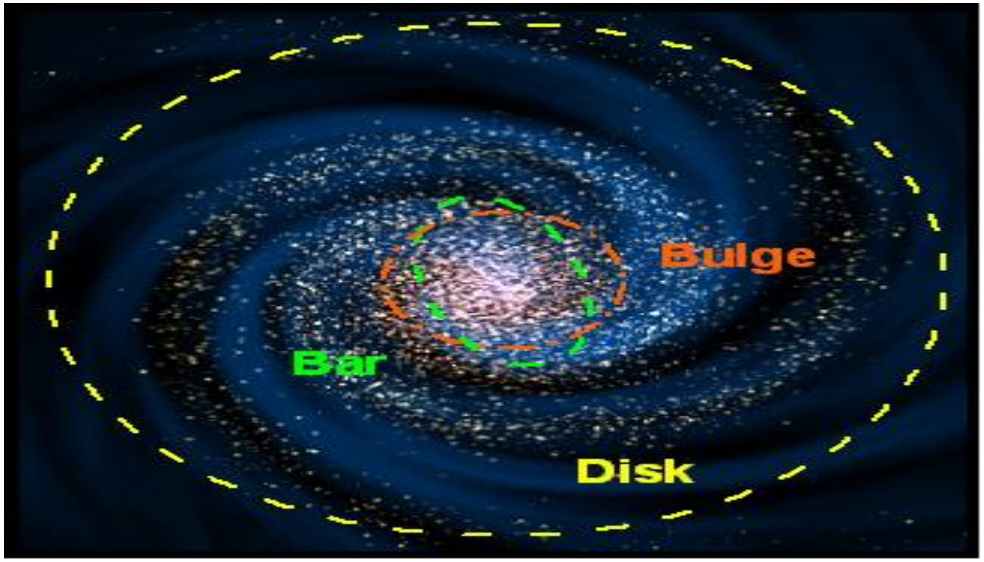


A time-dependent Hamiltonian system

Barred galaxies

NGC 1433

NGC 2217



Barred galaxy model

The 3D bar rotates around its short *z*-axis (*x*: long axis and *y*: intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z) - \Omega_b(xp_y - yp_x) \equiv Energy$$

This model consists of the superposition of potentials describing an axisymmetric part and a bar component of the galaxy (Manos, Bountis, Ch.S., 2013, J. Phys. A).

a) Axisymmetric component:

i) Plummer sphere:

$$V_{sphere}(x,y,z) = -\frac{GM_S}{\sqrt{x^2+y^2+z^2+\varepsilon_s^2}}$$

ii) Miyamoto–Nagai disc:

$$V_{disc}(x,y,z) = -\frac{GM_D}{\sqrt{x^2+y^2+(A+\sqrt{B^2+z^2})^2}}$$

b) Bar component:

$$V_{bar}(x,y,z) = -\pi Gabc\frac{\rho_c}{n+1}\int_\lambda^\infty \frac{du}{\Delta(u)}(1-m^2(u))^{n+1},$$

(Ferrers bar)

$$\rho_c = \frac{105}{32\pi}\frac{GM_B}{abc}$$

where $m^2(u) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}$, $\Delta^2(u) = (a^2+u)(b^2+u)(c^2+u)$,

n : positive integer ($n=2$ for our model) , λ: the unique positive solution of $m^2(\lambda)=1$

Its density is:

$$\rho = \begin{cases} \rho_c(1-m^2)^n, & \text{for } m \le 1 \\ 0, & \text{for } m > 1 \end{cases}, \text{ where } m^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2},\ a>b>c \text{ and } n=2.$$

Time-dependent barred galaxy model

The 3D bar rotates around its short *z*-axis (*x*: long axis and *y*: intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z,t) - \Omega_b(xp_y - yp_x) \equiv Energy$$

This model consists of the superposition of potentials describing an axisymmetric part and a bar component of the galaxy (Manos, Bountis, Ch.S., 2013, J. Phys. A).

a) Axisymmetric component:

$$M_S + M_B(t) + M_D(t) = 1, \quad \text{with} \quad M_B(t) = M_B(0) + \alpha t$$

i) Plummer sphere:

$$V_{sphere}(x,y,z) = -\frac{GM_S}{\sqrt{x^2 + y^2 + z^2 + \varepsilon_s^2}}$$

ii) Miyamoto–Nagai disc:

$$V_{disc}(x,y,z) = -\frac{GM_D(t)}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}$$

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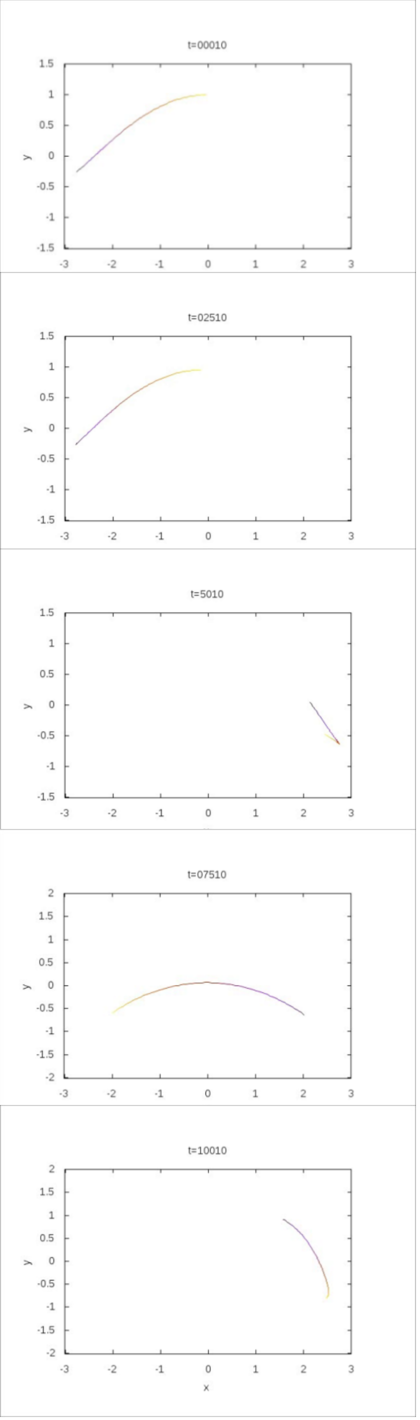
$$\text{where } m^2(u) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}, \quad \Delta^2(u) = (a^2+u)(b^2+u)(c^2+u),$$

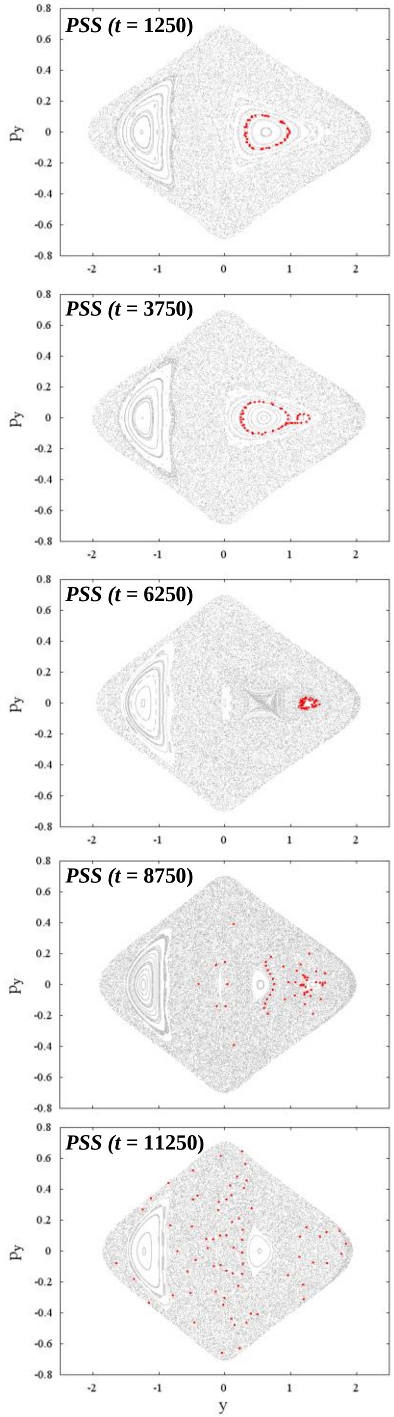
n: positive integer ($n = 2$ for our model), λ: the unique positive solution of $m^2(\lambda) = 1$

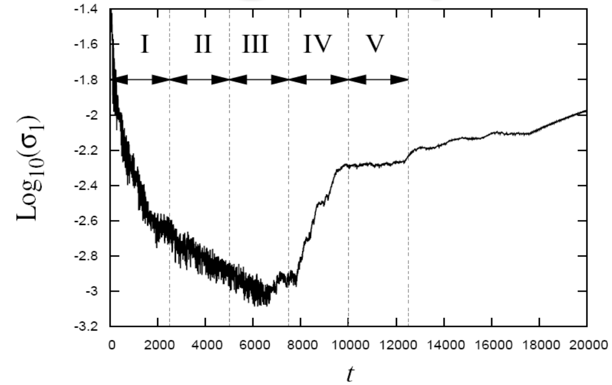
Its density is:

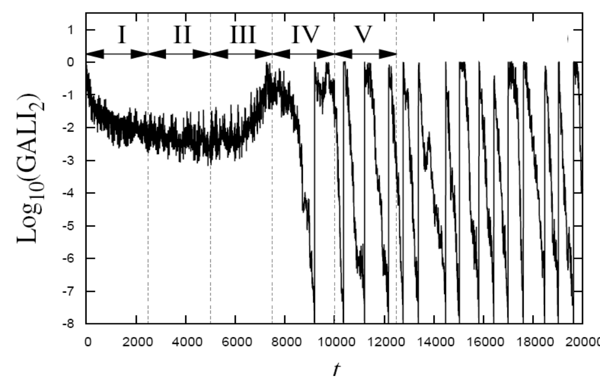
$$\rho = \begin{cases} \rho_c(1 - m^2)^n, & \text{for } m \le 1 \\ 0, & \text{for } m > 1 \end{cases}, \quad \text{where } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad a > b > c \text{ and } n = 2.$$

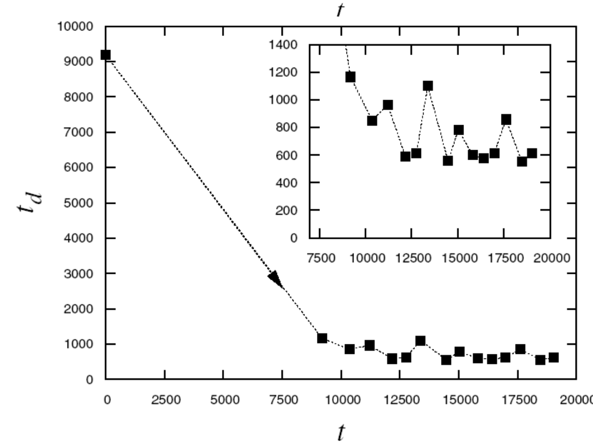
Time-dependent 2D barred galaxy model





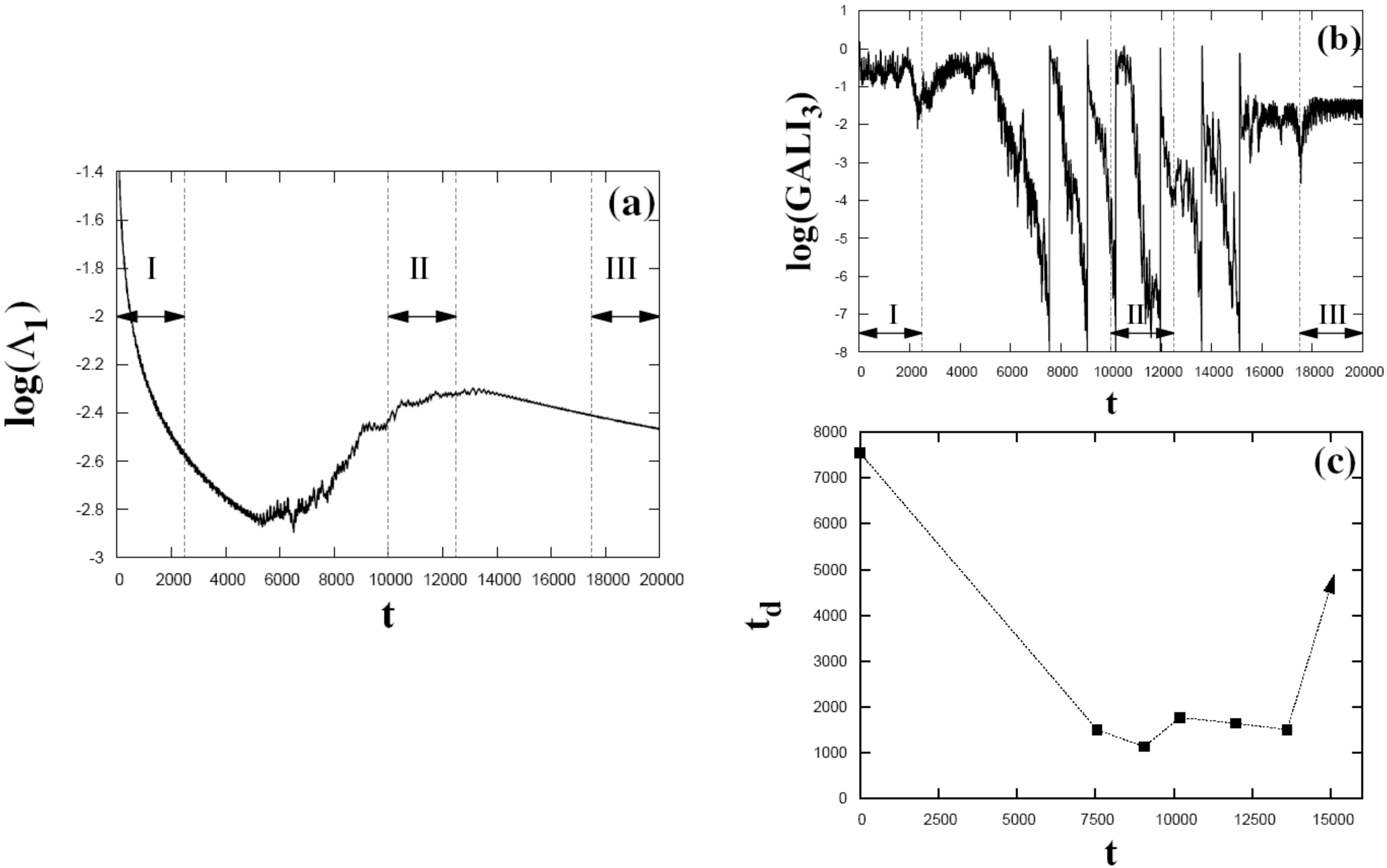






Time-dependent 3D barred galaxy model

Interplay between chaotic and regular motion



Numerical Integration of Equations of Motion and Variational Equations

Efficient integration of variational equations

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\vec{q},\vec{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

with $\vec{q} = (q_1(t), q_2(t), \ldots, q_N(t))$ $\vec{p} = (p_1(t), p_2(t), \ldots, p_N(t))$ **being respectively the coordinates and momenta.**

The time evolution of an orbit is governed by the Hamilton's equations of motion

$$\dot{\vec{q}} = \vec{p}$$

$$\dot{\vec{p}} = -\frac{\partial V}{\partial \vec{q}}$$

Variational Equations

The time evolution of a deviation vector

$$\vec{w}(t) = (\delta q_1(t), \delta q_2(t), \ldots, \delta q_N(t), \delta p_1(t), \delta p_2(t), \ldots, \delta p_N(t))$$

from a given orbit is governed by the variational equations:

$$\dot{\vec{\delta q}} = \vec{\delta p}$$
$$\dot{\vec{\delta p}} = -\mathbf{D}^2\mathbf{V}(\vec{q}(t))\vec{\delta q}$$

where $$\mathbf{D}^2\mathbf{V}(\vec{q}(t))_{jk} = \left.\frac{\partial^2 V(\vec{q})}{\partial q_j \partial q_k}\right|_{\vec{q}(t)}, \quad j,k = 1,2,\ldots,N.$$

The variational equations are the equations of motion of the time dependent tangent dynamics Hamiltonian (TDH) function

$$H_V(\vec{\delta q}, \vec{\delta p}; t) = \frac{1}{2}\sum_{j=1}^{N} \delta p_i^2 + \frac{1}{2}\sum_{j,k}^{N} \mathbf{D}^2\mathbf{V}(\vec{q}(t))_{jk}\delta q_j \delta q_k$$

Autonomous Hamiltonian systems

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Hamilton's equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

Integration of the variational equations

We use two general-purpose numerical integration algorithms for the integration of the whole set of equations:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \\ \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

a) the DOP853 integrator (Hairer et al. 1993, http://www.unige.ch/~hairer/software.html), which is an explicit non-symplectic Runge-Kutta integration scheme of order 8,

b) the TIDES integrator (Barrio 2005, http://gme.unizar.es/software/tides), which is based on a Taylor series approximation

$$\boldsymbol{y}(t_i + \tau) \simeq \boldsymbol{y}(t_i) + \tau \frac{\mathrm{d}\boldsymbol{y}(t_i)}{\mathrm{d}t} + \frac{\tau^2}{2!}\frac{\mathrm{d}^2\boldsymbol{y}(t_i)}{\mathrm{d}t^2} + \ldots + \frac{\tau^n}{n!}\frac{\mathrm{d}^n\boldsymbol{y}(t_i)}{\mathrm{d}t^n}$$

for the solution of system

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{y}(t))$$

Symplectic Integration schemes

Formally the solution of the Hamilton's equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \left\{H, \vec{X}\right\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n\geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where $\vec{X}$ is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} \approx \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B}$$

for appropriate values of constants c_i, d_i.

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.

Symplectic Integrator $SABA_2C$

We use a symplectic integration scheme developed for Hamiltonians of the form *H=A+εB* where A, B are both integrable and *ε* a parameter. The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator (Laskar & Robutel, 2001, Cel. Mech. Dyn. Astr.):

$$SABA_2 = e^{c_1 \tau L_A} \, e^{d_1 \tau L_{\varepsilon B}} \, e^{c_2 \tau L_A} \, e^{d_1 \tau L_{\varepsilon B}} \, e^{c_1 \tau L_A}$$

with $c_1 = \frac{(3-\sqrt{3})}{6}, \; c_2 = \frac{\sqrt{3}}{3}, d_1 = \frac{1}{2}$.

The integrator has only positive steps and its error is of order $O(\tau^4\varepsilon+\tau^2\varepsilon^2)$.

In the case where *A* is quadratic in the momenta and *B* depends only on the positions the method can be improved by introducing a corrector *C*={{A,B},B}, having a small negative step: $e^{-\tau^3 \varepsilon^2 \frac{c}{2} L_{\{\{A,B\},B\}}}$

with $c = \frac{(2-\sqrt{3})}{24}$.

Thus the full integrator scheme becomes: $SABAC_2 = C\,(SABA_2)\,C$ and its error is of order $O(\tau^4\varepsilon+\tau^4\varepsilon^2)$.

Tangent Map (TM) Method

Use symplectic integration schemes for the whole set of equations (Ch.S., Gerlach, 2010, PRE)

We apply the SABAC$_2$ integrator scheme to the Hénon-Heiles system (with ε=1) by using the splitting:

$$A = \frac{1}{2}(p_x^2 + p_y^2), \qquad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3,$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \{\{A, B\}, B\} = (x + 2xy)^2 + \left(x^2 - y^2 + y\right)^2$$

We approximate the dynamics by the act of Hamiltonians A, B and C, which correspond to the symplectic maps:

$$e^{\tau L_A}: \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases},$$

$$e^{\tau L_B}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1 + 2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases},$$

$$e^{\tau L_C}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}.$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

The system of the Hamilton's equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$\begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{aligned}$$

$$\begin{aligned} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y \end{aligned}$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

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\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}
\quad \xrightarrow{A(\vec{p})} \quad
\left.
\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= 0 \\
\dot{\delta p}_y &= 0
\end{aligned}
\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u}
$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

The system of the Hamilton's equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y
\end{aligned}$$

$$\begin{aligned}
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y
\end{aligned}$$

$A(\vec{p})$

$$\left.\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p_x} &= 0 \\
\dot{\delta p_y} &= 0
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u}$$

$B(\vec{q})$

$$\left.\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= 0 \\
\dot{\delta y} &= 0 \\
\dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u}$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

The system of the Hamilton's equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}$$

$A(\vec{p})$ →

$$\left.\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= 0 \\
\dot{\delta p}_y &= 0
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{AV}} : \left\{\begin{aligned}
x' &= x + p_x\tau \\
y' &= y + p_y\tau \\
px' &= p_x \\
py' &= p_y \\
\delta x' &= \delta x + \delta p_x\tau \\
\delta y' &= \delta y + \delta p_y\tau \\
\delta p_x' &= \delta p_x \\
\delta p_y' &= \delta p_y
\end{aligned}\right.$$

$B(\vec{q})$ →

$$\left.\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= 0 \\
\dot{\delta y} &= 0 \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{BV}} : \left\{\begin{aligned}
x' &= x \\
y' &= y \\
p_x' &= p_x - x(1+2y)\tau \\
p_y' &= p_y + (y^2 - x^2 - y)\tau \\
\delta x' &= \delta x \\
\delta y' &= \delta y \\
\delta p_x' &= \delta p_x - [(1+2y)\delta x + 2x\delta y]\tau \\
\delta p_y' &= \delta p_y + [-2x\delta x + (-1+2y)\delta y]\tau
\end{aligned}\right.$$

Tangent Map (TM) Method

So any symplectic integration scheme used for solving the Hamilton's equations of motion, which involves the action of Hamiltonians A, B and C, can be extended in order to integrate simultaneously the variational equations.

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p'_x &= p_x \\ p'_y &= p_y \end{cases}$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - x(1+2y)\tau \\ p'_y &= p_y + (y^2 - x^2 - y)\tau \end{cases}$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p'_y &= p_y - 2(y-3y^2+2y^3+3x^2+2x^2 y)\tau \end{cases}$$

Tangent Map (TM) Method

So any symplectic integration scheme used for solving the Hamilton's equations of motion, which involves the action of Hamiltonians A, B and C, can be extended in order to integrate simultaneously the variational equations.

$$e^{\tau L_A}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ p'_x &= p_x \\ p'_y &= p_y \end{cases} \longrightarrow e^{\tau L_{AV}}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x\tau \\ \delta y' &= \delta y + \delta p_y\tau \\ \delta p'_x &= \delta p_x \\ \delta p'_y &= \delta p_y \end{cases}$$

$$e^{\tau L_B}: \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - x(1+2y)\tau \\ p'_y &= p_y + (y^2 - x^2 - y)\tau \end{cases}$$

$$e^{\tau L_C}: \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p'_y &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2y)\tau \end{cases}$$

Tangent Map (TM) Method

So any symplectic integration scheme used for solving the Hamilton's equations of motion, which involves the action of Hamiltonians A, B and C, can be extended in order to integrate simultaneously the variational equations.

$$e^{\tau L_A}: \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases}$$

$$e^{\tau L_{AV}}: \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$$e^{\tau L_B}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases}$$

$$e^{\tau L_{BV}}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - \left[(1+2y)\delta x + 2x\delta y\right]\tau \\ \delta p_y' &= \delta p_y + \left[-2x\delta x + (-1+2y)\delta y\right]\tau \end{cases}$$

$$e^{\tau L_C}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}$$

Tangent Map (TM) Method

So any symplectic integration scheme used for solving the Hamilton's equations of motion, which involves the action of Hamiltonians A, B and C, can be extended in order to integrate simultaneously the variational equations.

$$e^{\tau L_A}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases} \rightarrow e^{\tau L_{AV}}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$$e^{\tau L_B}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases} \rightarrow e^{\tau L_{BV}}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - [(1+2y)\delta x + 2x\delta y]\,\tau \\ \delta p_y' &= \delta p_y + [-2x\delta x + (-1+2y)\delta y]\,\tau \end{cases}$$

$$e^{\tau L_C}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases} \rightarrow e^{\tau L_{CV}}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - 2\left[(1+6x^2+2y^2+6y)\delta x + \right. \\ & \left. +2x(3+2y)\delta y\right]\tau \\ \delta p_y' &= \delta p_y - 2\left[2x(3+2y)\delta x + \right. \\ & \left. +(1+2x^2+6y^2-6y)\delta y\right]\tau \end{cases}$$

Main References I

- **Hamiltonian systems and symplectic maps**
  - ✓ **Lieberman A. J. & Lichtenberg M. A. (1992) Regular and Chaotic Dynamics, Springer**
  - ✓ **Cvitanović P., Artuso R., Dahlqvist P., Mainieri R., Tanner G., Vattay G., Whelan N. & Wirzba A., (2015) Chaos – Classical and Quantum, version 15, http://chaosbook.org/**

- **Symplectic integrators and Tangent Map method**
  - ✓ **Ch.S., Gerlach E (2010) PRE, 82, 036704**
  - ✓ **Gerlach E, Ch.S. (2011) Discr. Cont. Dyn. Sys.-Supp., 2011, 475**
  - ✓ **Gerlach E, Eggl S, Ch.S. (2012) IJBC, 22, 1250216**
  - ✓ **Ch.S., Gerlach E, Bodyfelt J, Papamikos G, Eggl S (2014) Phys. Lett. A, 378, 1809**
  - ✓ **Gerlach E, Meichsner J, Ch.S. (2016) Eur. Phys. J. Sp. Top., 225, 1103**
  - ✓ **Senyange B, Ch.S. (2018) Eur. Phys. J. Sp. Top., 227, 625**
  - ✓ **Danieli C, Many Manda B, Mithun T, Ch.S. (2019) MinE, 1, 447**

Main References II

- The color and rotation (CR) method
  - ✓ Patsis P. A. & Zachilas L (1994) Int. J. Bif. Chaos, 4, 1399
  - ✓ Katsanikas M & Patsis P A (2011) Int. J. Bif. Chaos, 21, 467
  - ✓ Katsanikas M, Patsis P A & Contopoulos G. (2011) Int. J. Bif. Chaos, 21, 2321

- the 3D phase space slices (3PSS) technique
  - ✓ Richter M, Lange S, Backer A & Ketzmerick R (2014), Phys. Rev. E 89, 022902
  - ✓ Lange S, Richter M, Onken F, Backer A & Ketzmerick R (2014), Chaos 24, 024409
  - ✓ Onken F, Lange S, Ketzmerick R & Backer A (2016), Chaos 26, 063124

- Frequency Analysis
  - ✓ Laskar J (1990) Icarus, 88, 266
  - ✓ Laskar J, Froeschle C &Celletti A (1992) Physica D, 56, 253
  - ✓ Laskar J (1993) Physica D, 67, 257
  - ✓ Bartolini R, Bazzani A, Giovannozzi M, Scandale W & Todesco E (1996) Part. Accel. 52, 147
  - ✓ Laskar J (1999) in Hamiltonian systems with three or more degrees of freedom (ed. Simo C / Plenum Press) p 134

- Lyapunov exponents
  - ✓ Oseledec V I (1968) Trans. Moscow Math. Soc., 19, 197
  - ✓ Benettin G, Galgani L, Giorgilli A &Strelcyn J-M (1980) Meccanica, March, 9
  - ✓ Benettin G, Galgani L, Giorgilli A &Strelcyn J-M (1980) Meccanica, March, 21
  - ✓ Wolf A, Swift J B, Swinney H L &Vastano J A (1985) Physica D, 16, 285
  - ✓ Ch.S. (2010) Lect. Notes Phys., 790, 63

Main References III

- **0-1 test**
  - ✓ Gottwald G A & Melbourne I (2004) Proc. R. Soc. A, 460, 603
  - ✓ Gottwald G A & Melbourne I (2005) Physica D, 212, 100
  - ✓ Gottwald G A & Melbourne I (2009) SIAM J. Appl. Dyn., 8, 129
  - ✓ Gottwald G A & Melbourne I (2016) Lect. Notes Phys., 915, 221

- **FLI – OFLI – OFLI2**
  - ✓ Froeschle C, Lega E & Gonczi R (1997) Celest. Mech. Dyn. Astron., 67, 41
  - ✓ Guzzo M, Lega E & Froeschle C (2002) Physica D, 163, 1
  - ✓ Fouchard M, Lega E, Froeschle C & Froeschle C (2002) Celest. Mech. Dyn. Astron., 83, 205
  - ✓ Barrio R (2005) Chaos Sol. Fract., 25, 71
  - ✓ Barrio R (2006) Int. J. Bif. Chaos, 16, 2777
  - ✓ Lega E, Guzzo M & Froeschle C (2016) Lect. Notes Phys., 915, 35
  - ✓ Barrio R (2016) Lect. Notes Phys., 915, 55

- **MEGNO**
  - ✓ Cincotta P M & Simo (2000) Astron. Astroph. Suppl. Ser., 147, 205
  - ✓ Cincotta P M, Giordano C M & Simo C (2003) Physica D, 182, 151
  - ✓ Cincotta P M, & Giordano C M (2016) Lect. Notes Phys., 915, 93

- **RLI**
  - ✓ Sandor Zs, Erdi B & Efthymiopoulos C (2000) Celest. Mech. Dyn. Astron., 78, 113
  - ✓ Sandor Zs, Erdi B, Szell A & Funk B (2004) Celest. Mech. Dyn. Astron., 90 127
  - ✓ Sandor Zs & Maffione N (2016) Lect. Notes Phys., 915, 183

Main References IV

- **SALI**
  - ✓ Ch.S. (2001) J. Phys. A, 34, 10029
  - ✓ Ch.S., Antonopoulos Ch, Bountis T C & Vrahatis M N (2003) Prog. Theor. Phys. Supp., 150, 439
  - ✓ Ch.S., Antonopoulos Ch, Bountis T C & Vrahatis M N (2004) J. Phys. A, 37, 6269
  - ✓ Bountis T & Ch.S. (2006) Nucl. Inst Meth. Phys Res. A, 561, 173
  - ✓ Boreaux J, Carletti T, Ch.S. &Vittot M (2012) Com. Nonlin. Sci. Num. Sim., 17, 1725
  - ✓ Boreaux J, Carletti T, Ch.S., Papaphilippou Y & Vittot M (2012) Int. J. Bif. Chaos, 22, 1250219

- **GALI**
  - ✓ Ch.S., Bountis T C & Antonopoulos Ch (2007) Physica D, 231, 30
  - ✓ Ch.S., Bountis T C & Antonopoulos Ch (2008) Eur. Phys. J. Sp. Top., 165, 5
  - ✓ Gerlach E, Eggl S & Ch.S. (2012) Int. J. Bif. Chaos, 22, 1250216
  - ✓ Manos T, Ch.S. & Antonopoulos Ch (2012) Int. J. Bif. Chaos, 22, 1250218
  - ✓ Manos T, Bountis T & Ch.S. (2013) J. Phys. A, 46, 254017

- **Reviews on SALI and GALI**
  - ✓ Bountis T C & Ch.S. (2012) 'Complex Hamiltonian Dynamics', Chapter 5, Springer Series in Synergetics
  - ✓ Ch.S. & Manos T (2016) Lect. Notes Phys., 915, 129